SIX LECTURES

ON

THE MEAN-VALUE THEOREM

OF

THE DIFFERENTIAL CALCULUS

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ON
THE MEAN-VALUE THEOREM
OF
THE DIFFERENTIAL CALCULUS

delivered at the Calcutta University

BY

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PREFACE

In this book are contained without any material alteration the six public lectures which were delivered by me during the first four months of 1930 with the two-fold object of (1) giving an up-to-date account of our knowledge of the mean-value theorem and the function θ to which I have ventured to give the name of Rolle's function, and (2) stimulating research relating to Rolle's function by suggesting problems which were at the time engaging my attention and had not been completely solved.

Apart from the first chapter, which is historical and introductory, the book may be roughly divided into two parts, *viz.*, the second and third chapters which deal with the mean-value theorem and its generalizations, and the fourth and fifth chapters and the major part of the sixth chapter in which Rolle's function has been studied. The first Appendix is of interest because of the correspondence between Professor Pompeiu and myself about his remarkable proof of the mean-value theorem. The second Appendix deals with the history of the various forms of the remainder in Taylor's series. The third Appendix contains corrections and additions.

I venture to say that the chief interest of the book is the prominent place given in it to Rolle's function. It is true that the function was considered by Cauchy about 100 years ago and was later studied by American and Cambridge mathematicians without more being discovered about its functional property than that it may be multiple-valued. It is only very recently that well-known mathematicians, for example, Professor Rudolf Rothe and Professor T. Hayashi, took up the study of the function but in ignorance of much of the earlier work. A few of the landmarks in this field of research may be enumerated here. (1) Professor Rothe had never contemplated the possibility of $\theta(h)$ being non-differentiable; I have given functions $\theta(h)$ which are single-valued, finite and continuous and at the same time without differential co-efficients at the points of an everywhere dense set. (2) Prof. Hedrick had attempted the study of $\theta(h)$ as a multiple-valued function, but, for want of the separate treatment of the different values corresponding to a given h, his treatment is confused and infructuous. I have introduced the notion of *principal value* of $\theta(h)$ as the greatest of the different values and shown

that, for $f'(h)$ as nowhere differentiable, $\theta(h)$ may be also nowhere differentiable, with the possible exception of the points of the first category where $f'(h)$ has "cusps."

It is a great pleasure to me to record my obligations to a number of my friends and pupils who have helped me in various ways, during the time the book was in the Press. To Dr. Bibhutibhushan Datta, D.Sc., I am indebted for his advice relating to the parts which are of historical character. Dr. A. N. Singh, D.Sc., Lecturer in the Lucknow University, has gone through nearly the whole of the book in proof and has given me valuable advice relating to certain parts of it. Mr. Hariprasanna Banerjee, M.Sc., Lecturer in Pure Mathematics to the Post-graduate students in the Calcutta University, has also helped me like Dr. Singh with valuable advice. Some of my present researchers, specially Mr. Bholanath Mookerjee, M.A., P.R.S., of Scottish Church College, Mr. Santoshkumar Bhar, M.Sc., and Mr. Rama Dhar Misra, M.A., have also given me help in going through the proof-sheets.

CALCUTTA:

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GANESH PRASAD

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FIRST LECTURE

Historical and Introductory.

Colleagues, students and other gentlemen!

The six lectures which I propose to deliver on the mean-value theorem, rightly called the fundamental theorem of the Differential Calculus, are intended to make known to you a field of research which has only in comparatively recent times attracted anything like considerable attention from prominent mathematicians and which is far from being exhausted. I shall feel amply rewarded if I succeed in inspiring some of you with enthusiasm to take up for research even a few of the numerous problems that may be suggested by my lectures.

§ 1.

1. To-day's lecture will be of a historical and introductory character, and I shall begin by giving you an account of the vicissitudes through which Rolle's theorem, from which the mean-value theorem originated, has gone. Michel Rolle (1652-1719), who was from 1685 onwards a paid member of the Academy of Sciences of Paris, was one of the small band of the "old mathematicians who did not content themselves with the intuitional proofs of theorems but demanded rigorous logical definitions and proofs."[1] In a small duodecimo book,[2] published in 1691, he gave what goes now by the name of Rolle's theorem,[3] viz., $f'(x)=0$ has at least one

[1] See Felix Klein's *Anwendung der Differential- und Integralrechnung auf Geometrie*, 1st edition, p. 114.

[2] *Démonstration d'une Méthode pour résoudre les Egalitez de tous les degrez ; suivie de deux autres Méthodes, dont la premiere donne les moyens de résoudre ces mêmes égalitez par la Géométrie et la seconde, pour resoudre plusieurs questions de Diophante qui n'ont pas encore esté resolues*, Paris 1691.

[3] In his *Traité d'algèbre* (1690), Rolle gave the "method of cascades." By a cascade of an equation

$$f(x)=a_0+a_1x+\dots+a_{n-1}x^{n-1}+a_nx^n=0$$

being understood the equation obtained by multiplying the terms of the original equation by the terms of the progression 0, 1, 2, ..., n and dividing both sides by x; in other words the equation $f'(x)=0$. The theorem on which the method of cascades is based is not Rolle's theorem but a corollary to it, viz., between two successive real roots of $f'(x)=0$ there cannot be more than one real root of $f(x)=0$.

[illegible] lying between two [illegible]. Statements[1] which [illegible] theorem are found in [illegible] which appeared [illegible] until the [illegible] in 1797 the [illegible] the theorem [illegible] Lagrange [illegible] which appeared in [illegible]. But, according to Professor Cajori,[2] [illegible] we have cited [illegible] directly [illegible] the theorem [illegible] stand out conspi[illegible] as a theorem, [illegible] in passing. This attitude [illegible] with many writers of the first half of the nineteenth century. [illegible] De Morgan's *Differential and Integral* [illegible] in 1842 [illegible] also absent [illegible] Differential Calculus [illegible] in the [illegible] by R. [illegible] In [illegible] theory of equations [illegible] In [illegible] in the German [illegible] of Serret's *Algebra* [illegible] then the theorem [illegible] be found [illegible] we see the [illegible] theory of equa[illegible] in the wider re[illegible] mathematical [illegible] first magnitude. Rolle's theo[illegible] from which radi[illegible] make the modern develop[illegible] of the [illegible].

§ 2.

2. A [illegible] theorem [illegible] form

$$f(x+h) = f(x) + [\text{illegible}] \qquad (1)$$

appeared for the first time in [illegible] 182[illegible] it was given in

[1] [illegible] Works, Series [illegible]

[2] [illegible] book *Méthode pour résoudre les égalités* and the history of Rolle's theorem [illegible] Mathematics [illegible] Vol. XI [illegible] pp. [illegible].

nearly the [illegible] form by [illegible] who first [illegible] the remainder in [illegible] expansion for [illegible] after [illegible] terms and then put [illegible] [illegible] Pringsheim[1] recognises the two inequalities

$$\left.\begin{array}{l}\phi(x+h)-\phi(x)-h\,\phi'(x+p)>0,\\ \phi(x+h)-\phi(x)-h\,\phi'(x+[illegible])\end{array}\right\}\ (0<p\leq h)\qquad (2)$$

where $\phi'(x+p)$ and [illegible] are respectively the minimum and maximum of $\phi'(x+$[illegible]$)$ for $0<$[illegible]$\leq h$ as equivalent to the [illegible] theorem. [illegible] appear in [illegible] [illegible] generalised form and in the [illegible] form in [illegible] (1806) and in Ampère's paper [illegible] [illegible] printed in the [illegible] des [illegible] [illegible] year Cauchy gives these [illegible] in his [illegible] and [illegible] form [illegible] at [illegible] Paris. Perhaps the first [illegible] together the theorem of mean value and [illegible] theorem was [illegible] derivative [illegible] the theorem of mean [illegible] given by Serret[2] in his [illegible] (1868). Serret [illegible] however [illegible] the name of Bonnet in the proof[3] [illegible] under which [illegible] to appear [illegible] in the works of the writers who followed [illegible] The present [illegible] for the validity of (1) is [illegible] first given with [illegible] by Dini in his lectures[4] at Pisa during the years 1871-1872 [illegible] and proof [illegible] both Bonnet's proof and Jordan's proof [illegible] first [illegible] of his Cours d'Analyse, Vol. I, of 1882. The conditions for the validity of (1) were made less restrictive by Prof. W. H. Young and Dr. [illegible] Young[5] and still less restrictive by Dr. A. N. Singh.

4. I proceed now to give a brief historical account of [illegible] of θ which seems to have been first studied by Cauchy and [illegible]

[1] "Zur Geschichte des Taylorschen Satzes," [illegible] 3, No. 3, 1900, pp. 433-479, specially pp. 444-450.

[2] [illegible] in the shape of the book [illegible] di variabili reali.

[3] [illegible] for 1884, pp. [illegible] Jordan's proof referred to is the one which appeared [illegible] 1882.

[4] [illegible] and the theorem of the mean [illegible] Quarterly Journal of Mathematics, Vol. XL, 1909, pp. 1-26.

[5] On the [illegible] theorem of the Differential Calculus [illegible] Mathematical Society, Vol. XIX, [illegible] pp. 48-49.

[illegible] ed only in the last nine years.

§ 3

[illegible]

$$f(x_0+h)-f(x_0)=hf'(x_0+\theta h),$$

[illegible]

[illegible]

(Bulletin of the Cal. Math. Soc., Vol. 19, pp. 143-160)

[illegible]

Hayashi, Science Reports, Tohoku Imp. Univ., 1923.

[illegible]

[illegible]

$$\phi(b)-\phi(a)=(b-a)\,\phi'(a+\theta\,\overline{b-a}),\ \theta<1$$

III. [illegible] If [illegible] greater than [illegible] and [illegible] be a function of x which [illegible] and continuous for all values of x between [illegible] and [illegible] then

$$F(x_n)-F(x_0)=(x_n-x_0)\,F'(x_0+\theta\,\overline{x_n-x_0}),$$

θ being some *proper positive fraction.*

IV. [illegible]

V. [illegible] and without [illegible] values from [illegible] differential coefficient [illegible] the [illegible] then

$$\phi(a+h)-\phi(a)=h\phi'(a+\theta h)$$

[illegible] of θ [illegible]

VI. [illegible] when x passes from x_0 to x_0+h then

$$f(x_0+h)-f(x_0)=hf'(x_0+\theta h),$$

θ being a number comprised between [illegible] and 1. [illegible] in value the term is first given and the above is deduced from it.

VII. [illegible] When the function $f(x)$ has a single value for [illegible] and remains continuous [illegible] differential coefficient $f'(x)$, from $x=x_0$ up to $x=x_0+h$ [illegible] there exists between the limits 0 and 1 a value of θ such that

$$f(x_0+h)-f(x_0)=hf'(x_0+\theta h).$$

[illegible]

[illegible] VIII. [illegible] The mean-value theorem [illegible] explicitly [illegible] the Lagrange's remainder [illegible] formula [illegible] mean-value theorem [illegible] the following. If f, together with its differential [illegible]

$$f(X)-f(x_0)=(X-x_0)f'\{x_0+\theta(X-x_0)\},\ \theta<1$$

IX. [illegible] and [illegible] in an interval [illegible] in the interval [illegible] progressive differential co-efficient $f'(x+0)$, then

$$f(x_0+h)-f(x_0)=hf'(\xi+0),$$

ξ being between x_0 and x_0+h.

X. [illegible] Same as [illegible] by [illegible] with the difference that [illegible] the inequality $0\leq\theta\leq1$.

XI. [illegible] If [illegible] in the whole [illegible] and [illegible] the ends of the interval, [illegible] points [illegible] differential [illegible] which is finite [illegible] determinate in sign, then [illegible] positive or negative [illegible] points whatever of this interval [illegible] and by θ a number comprised between [illegible] which depends on x and on h we have always

$$f(x+h)=f(x)+hf'(x+\theta h)$$

§ 4

[illegible] Prof. W. H. Young and [illegible] C. Young [illegible] generalisation of the mean-value theorem [illegible] that the absence of the differential co-efficient at any point inside the interval $(x, x+h)$ might render invalid the relation

$$f(x+h)=f(x)+hf'(x+\theta h),\ 0<\theta<1.$$

This point has been carefully elaborated in Schlömilch's *Höhere Analysis* [illegible] 1, 1881. The first investigators to relax the conditions for the

[illegible] *Analyse* [illegible] pp. [illegible] 50.

[illegible] *Intégrale*, p. 10. Note that the condition of the [illegible] $f'(x+0)$ [illegible] existence and continuity of $f(x)$.

[illegible] p. 165.

[illegible] p. 75.

Loc. [illegible]

existence of the differential coefficient were the two Youngs. Their generalization is the following:

XII. If there is no [illegible] of right and left with regard to the derivates of $f(x)$, then there is a point in the completely open interval (a, b) at which $f(x)$ has a differential coefficient and the value of that differential coefficient is precisely

[illegible]

[illegible]

The special case in which [illegible] is [illegible] the generalised Rolle's theorem and the following [illegible]. If $f(x)$ is a finite function [illegible] has [illegible] only at [illegible] where [illegible] and is further continuous throughout the closed interval (a, b), and [illegible] at the end points a, b, then there is a point ξ of the completely open interval (a, b) at which one of the upper derivates is not positive and the other lower derivate is not negative.

$$\text{i.e.,} \quad f^{+}(\xi) \leq 0 \leq f_{-}(\xi),$$

or the alternative inequality interchanging left and right,

$$f^{-}(\xi) \leq 0 \leq f_{+}(\xi)$$

7. Singh[1] has gone further than the Youngs and his generalisation [illegible] [illegible] the [illegible] in the derivates. Singh's generalisation, is the following:

XIII. If $f(x)$ be a continuous function defined in the closed interval (a, b), such that

(1) there is no point within (a, b) at which one of the derivates [illegible] progressive differential coefficient or regressive differential coefficient) exists while the other does not exist, and [illegible] at each point within (a, b) the upper and lower derivates on one side are either [illegible] equal to the upper and lower derivates on the other side, then there exists a point in the completely open interval (a, b) at which the differential coefficient exists and its value is equal to

[illegible]

Singh calls the following theorem [illegible] a theorem for derivatives: If $f(x)$ be a continuous function defined in the interval [illegible] that the upper

[illegible] the upper and [illegible] completely [illegible] this point and its value is

$$\frac{[illegible]}{b-a}$$

§ 5.

[illegible]

$$\theta = \tfrac{1}{2} + hA_1 + h^2A_2 + \dots + h^nA_n + \dots \text{ to infinity}$$

[illegible]

[illegible] the [illegible] function θ [illegible] order that [illegible] should [illegible]

§ 6

10. The fact that θ may be a [illegible] [illegible] [illegible] [illegible] [illegible] fact [illegible] later [illegible] [illegible] the function [illegible] function [illegible] multiple-valued function. [illegible] the principal value [illegible] [illegible] [illegible] Arranging [illegible] of magnitude [illegible] shown to hold for [illegible]

11. [illegible] variables x and h, and [illegible]

(a) [illegible] $a \leq x \leq$ [illegible] $x+h \leq$ [illegible] [illegible] then [illegible] theorem holds [illegible] independent of x and h [illegible]

(b) With the assumptions made in (a) [illegible] of x and dependent only on h, then [illegible] value theorem is

$$\theta = \frac{1}{ah}\log\frac{e^{ah}-1}{ah}$$

and the corresponding [illegible]

[illegible] multiple-valued function [illegible]

* "On Rolle's function θ as multiple-valued function," [illegible] *Mathematical Society*, Vol. [illegible] [illegible]

[illegible] With the supposition [illegible] $f(x+h)$ [illegible] independent of h and depends only on x, [illegible] which also possesses

[illegible]

In the [illegible] paper the [illegible] that there should [illegible] function [illegible] theorem

$$f(x+h) = f(x) + h\, f'(x + \theta h)?$$

In treating this question [illegible] the [illegible] forms of

$$\theta_1 = [illegible], \quad \dots, \quad \theta_{1\,2} = \frac{\partial\theta}{\partial x_1}\,[illegible], \quad \theta_{1\,2\,2} = \frac{\partial\theta_{1\,2}}{\partial x_2}$$

where $x_1 = x$ and $x_2 = x + h$.

§ 7.

12. [illegible] miscellaneous theorems [illegible] connected with the mean value theorem [illegible] general [illegible] mean value theorem

$$[illegible] \qquad 0 < \theta_1 < 1$$

In [illegible] known [illegible] of the above theorem [illegible] and [illegible] may be [illegible]

[illegible]

$$\begin{vmatrix} f(x+h) & \phi(x+h) & \psi(x+h) \\ f(x) & \phi(x) & \psi(x) \\ f(x+\theta h) & \phi(x+\theta h) & \psi(x+\theta h) \end{vmatrix} = 0, \quad 0 < \theta < 1$$

See [illegible] differential and integral [illegible]. For [illegible] $= 1$, the above reduces to Cauchy's generalized mean value theorem.

[illegible] Pompeiu,[1] T. Hayashi,[2] and Takahashi [illegible] that under certain conditions for x and h [illegible] $(0<\theta_n<1)$ [illegible]

The [illegible] of Hayashi and Takahashi [illegible]

[illegible] The next group of results [illegible] mentioned [illegible] relating to the [illegible] Lagrange's remainder [illegible] $x+\theta_n h$ [illegible] mean value theorem [illegible] theorem. [illegible] gave the expansion [illegible] the coefficients of [illegible] $\frac{1}{n+1}$. The function θ_n has been studied by [illegible] and the [illegible] results, among others, have been [illegible] by him.

[illegible] In order that [illegible] $f(x)$ satisfying Taylor's theorem with the remainder [illegible] θ_n [illegible] of x and h [illegible] and sufficient that $f(x)$ is a polynomial of the $(n+1)$th degree, and that $\theta_n = \frac{1}{n+1}$.

(b) $\lim_{h=+0} \theta_n = \frac{1}{n+1}$ if $f^{(n+1)}(x)$ is existent and continuous for [illegible] $< x < h$ and at the same time [illegible]

[illegible] $\frac{1}{n+1}\left\{\frac{1}{n+2}-\frac{1}{2(n+1)}\right\} \cdot \frac{f^{(n+2)}(x)}{f^{(n+1)}(x)}$ [illegible]

certain conditions. The work of [illegible] in this connection is open to the same criticism as in the case of [illegible] function θ.

[1] Sur [illegible] théorème des accroissements finis [illegible] 1925, pp. 61-64].
[2] [illegible] Report [illegible] Vol. 15, 1926, pp. 185-191.
[3] [illegible] Vol. 30, 1929, p. 134.
[4] [illegible] pp. 349-371.
[5] [illegible] Zeit. [illegible] pp. [illegible]
[6] Math. Zeit. Bd. [illegible] pp. 305-31[illegible]
[7] Tohoku M[illegible] J., Vol. 29, 192[illegible], p. 151.

[illegible]

[illegible]

$$f(x+h, y+k) - f(x, y)$$

(a) The result that

[illegible]

(b) The result that

[illegible]

$$x = x_0 + ht, \qquad y = y_0 + kt.$$

[illegible]

SECOND LECTURE

Proofs of the Mean value Theorem

[illegible] almost word by word.

[illegible]

[illegible]

x_1 being a value comprised between [illegible] and X.

In fact, the ratio

[illegible]

[illegible] have

$$\{f(X) - AX\} - \{f(x_0) - Ax_0\} = 0. \tag{1}$$

Let us [illegible] the function [illegible]

$$\phi(x) = \{f(x) - Ax\} - \{f(x_0) - Ax_0\}, \tag{2}$$

then we shall have because of the equality (1)

[illegible]

so that $\phi(x)$ vanishes for $x = x_0$ and for $x = X$.

Serret [illegible] Differential [illegible]

[illegible]

[illegible] difference

$$\phi(x_1 - h) - \phi(x_1), \quad \phi(x_1 + h) - \phi(x_1)$$

[illegible]

$$\frac{\phi(x_1 - h) - \phi(x_1)}{-h}, \quad \frac{\phi(x_1 + h) - \phi(x_1)}{h} \qquad \text{(ii)}$$

will be of opposite signs.

[illegible]

One has therefore

$$\frac{f(X)-f(x_0)}{X-x_0}=f'(x_1), \qquad (4)$$

as was announced.

We have supposed that $X > x_0$ but [illegible] preceding [illegible] not change with [illegible] of this hypothesis.

If one puts

$$X = x_0 + h$$

the quantity x_1 comprised between x_0 and X [illegible] $x_0 + \theta h$, θ being a quantity comprised between 0 and 1 [illegible] write

$$f(x_0+h)-f(x_0)=hf'(x_0+\theta h).$$

It [illegible] is necessary to make a few remarks about the above proof.

(1) Although the function [illegible] for the values of x comprised between the two limits x_0 and X and not at the extremities [illegible] shall attain its upper and [illegible] lower bound at a [illegible] point [illegible] because the [illegible] limits will not necessarily be [illegible]

(2) The proof does not assume the [illegible] of X.

(3) The proof does not assume either the [illegible] any where or its finiteness.

(4) The proof is [illegible] as pointed [illegible] because of the statement "If one admits that [illegible] for the values of x comprised between x_0 and X [illegible] taking positive values [illegible] by taking negative values [illegible] by starting from [illegible] comprised between x_0 and X."

If we take $\phi(x)$ to be [illegible] although $\phi(x_0)=0$ it cannot be said that [illegible] because in any interval ever so small [illegible] number of fluctuations.

[illegible]

§ 9

[illegible]

[illegible] between x_n and x_0, then

$$F(x_n) - F(x_0) = (x \ldots$$ [illegible]

θ being some proper positive fraction.

[illegible] $x_1 - x_0$, $x_2 - x_1$, … $x_n - x_{n-1}$, is an infinitesimal. [illegible] following series of equations:

$$\left.\begin{aligned} F(x_1) - F(x_0) &= (x_1 - x_0)F'(x_0), \\ F(x_2) - F(x_1) &= (x_2 - x_1)F'(x_1) \\ &\;\;\vdots \\ F(x_n) - F(x_{n-1}) &= (x_n - x_{n-1})F'(x_{n-1}), \end{aligned}\right\} \qquad (1)$$

whence [illegible] of the second factors, that is

[illegible]

[illegible]

§ 10

14. In [illegible] proof given [illegible]

[illegible] pp. 44-45.

If the fun[illegible] A [illegible] difference [illegible] $f($ $)$ [illegible]

$$\frac{f(X)-f(x_0)}{X-x_0} \quad \ldots \quad (4)$$

[illegible] and B.

Let us [illegible] by [illegible] in such a manner that [illegible] any value whatever [illegible]

$$\frac{f(x+i)-f(x)}{i}$$

remains always greater than [illegible] between the [illegible] x_0, X [illegible]

$$x_1, x_2 \ldots$$

so as to divide the difference $X-$ [illegible] into [illegible]

$$x_1-x_0, \; x_2-x_1 \ldots X$$

which [illegible] the fractions

$$\frac{f(x_1)-f(x_0)}{x_1-x_0}, \; \frac{f(x_2)-f(x_1)}{x_2-x_1}, \ldots \frac{f(X)-\ldots}{X-\ldots} \quad (5)$$

being found [illegible] the first [illegible] on the [illegible] quantity A [illegible] and less than the quantity B [illegible] tions (5) having [illegible] the numerators [illegible] fraction that is [illegible] those which are considered.

The expression [illegible] therefore enclosed between the limits A and B [illegible] however small [illegible] the number [illegible] also be comprised between A and B.

Corollary. [illegible] between the limits [illegible] X [illegible] this function shall vary in such a manner [illegible]

[illegible] between A and B and [illegible] take [illegible] the intermediate values [illegible] therefore every [illegible] quantity [illegible] A and B [illegible] be a value [illegible] corresponding to a value of [illegible] between [illegible] and $X = x_0 + h$ or what [illegible] to the same thing [illegible]

$$x_0 + \theta h = x_0 + \theta\,(X - x_0),$$

θ denoting a number less than 1."

[illegible] remarks [illegible] proof.

[illegible] and [illegible] in the [illegible] necessary for the proof that

$$\frac{f(X) - f(x_0)}{X - x_0}$$

lies between A and B.

(b) [illegible] quantities that [illegible] X the ratio [illegible] greater than [illegible] and less than [illegible] that [illegible] for every value of x. [illegible] to the continuity of $f'(x)$.

[illegible] it is evident [illegible] for every value of [illegible] the [illegible] Let [illegible] of the discontinuity [illegible] uniformly tend to [illegible] point [illegible] taking [illegible]

$$\frac{1}{[illegible]} \quad \frac{1}{[illegible]} \quad \frac{1}{[illegible]}$$

where m is integral [illegible] so that the difference [illegible] is numerically equal to 1 and is not a vanishing quantity with increasing m.

11

21. Another fairly careful proof [illegible] De Morgan [illegible] below. Let there be two limits [illegible] and $a + h$ such that [illegible] for [illegible]

nor between them [illegible] any [illegible]. Thus for [illegible] from [illegible] 2 to [illegible] there [illegible] of log [illegible] of them singular. We have now P [illegible] whatever the [illegible]. Consequently P and [illegible] will [illegible] remain [illegible] vary in any manner between a and a + h. Thus for instance [illegible] and [illegible] are [illegible] even [illegible] should vary in any manner between a and a + h.

Let [illegible] suppose [illegible] to be the [illegible] part [illegible] that [illegible] without [illegible] P which is a function of x and [illegible] be [illegible] and we then have

$$\frac{\phi(x+\Delta x)-\phi(x)}{\Delta x}=\phi'(x)+f(x, \Delta x);$$

now [illegible] to have $\phi(x+n\Delta x)$ [illegible] in the numerator, which will give the following set of equations (n in number):—

$$\frac{\phi(x+\Delta x)-\phi(x)}{\Delta x}=\phi'(x)+f(x, \Delta x),$$

$$\frac{\phi(x+2\Delta x)-\phi(x+\Delta x)}{\Delta x}=\phi'(x+\Delta x)+f(x+\Delta x, \Delta x),$$

$$\frac{\phi(x+3\Delta x)-\phi(x+2\Delta x)}{\Delta x}=\phi'(x+2\Delta x)+f(x+2\Delta x, \Delta x),$$

$$\cdots$$

$$\frac{\phi(x+\overline{n-1}\,\Delta x)-\phi(x+\overline{n-2}\,\Delta x)}{\Delta x}=\phi'(x+\overline{n-2}\,\Delta x)+f(x+\overline{n-2}\,\Delta x, \Delta x),$$

$$\frac{\phi(x+n\Delta x)-\phi(x+\overline{n-1}\,\Delta x)}{\Delta x}=\phi'(x+\overline{n-1}\,\Delta x)+f(x+\overline{n-1}\,\Delta x, \Delta x).$$

Form the fraction which has the sum of the numerators of the preceding for its numerator and the sum of the denominators for its denominator, it being clear that all the denominators have the same sign.

[1] For the definition of the expression [illegible] see p. [illegible] of De Morgan's book. The definition is, however, not clear [illegible] that can be safely assumed is that De Morgan meant [illegible] to have a singular value at [illegible] of $\phi(x)$ had a discontinuity at [illegible].

[2] P is said to be a convergent if it tends to [illegible] as [illegible] tends to zero, e.g. [illegible]

That given

[illegible]

$$\phi'(a + h\triangle x) = f(a + h\triangle x, \triangle x)$$

[illegible]

know that

[illegible]

$$\phi'(a) + f(a + h'\triangle x, \triangle x),$$

[illegible]

[illegible] lies between [illegible]

[illegible]

*See §§ 15 and 16 of De Morgan's book

$\varphi'(\ )$ between $\varphi'(t)$ and $\varphi'(\ )$, and since $a + \theta h$ [illegible] between 0 and 1, ...

$$\frac{\varphi(a+h)-\varphi(a)}{h}=\varphi'(a+\theta h)."$$

§ 12

21. The following proof, taught by [illegible], may be considered [illegible] a precursor of [illegible] proof [illegible] the generalized mean-value theorem of Cauchy [illegible]. Let $f(x)$ and $F(x)$ be two real functions which vanish for $x = x_0$, and which remain continuous between the limits $x = x_0$, $x = X$. Let us suppose that the differential coefficient $F'(x)$ does not change its sign between the limits in question. If one calls A the least and B the greatest of the values which the ratio

$$\frac{f'(x)}{F'(x)}$$

receives in this interval, then the fraction

$$\frac{f(x)}{F(x)}$$

shall also remain comprised between the two limits A and B.

Proof. "Because one shall have, by hypothesis, for all the values of x enclosed between the limits x_0 and X,

$$\frac{f'(x)}{F'(x)}-A>0,\quad \frac{f'(x)}{F'(x)}-B<0$$

and because the differential coefficient $F'(x)$ does not change its sign between these limits, one may affirm that each one of the products

$$F'(x)\left\{\frac{f'(x)}{F'(x)}-A\right\}=f'(x)-AF'(x),$$

$$F'(x)\left\{\frac{f'(x)}{F'(x)}-B\right\}=f'(x)-BF'(x)$$

Leçons sur le calcul différentiel (*Œuvres*, Series 2, t. IV, pp. 4[illegible]). This proof was first given in the *Addition* to the *Résumé* (*Œuvres*, Series 2, t. IV, pp. 24[illegible]).

[illegible] the [illegible] Moreover these products are [illegible] the functions

$$f(x)-AF(x),\ f(x)-BF(x).$$

[illegible]

$$f(X)-AF(X),\ f(X)-BF(X)$$

[illegible]

$$\frac{f(X)}{F(X)}-A,\ \frac{f(X)}{F(X)}-B$$

[illegible]

[illegible] If the [illegible] are themselves [illegible] $x=X$, then when one passes [illegible] shall vary in such a manner [illegible] A and B [illegible] Therefore every mean quantity [illegible] shall be a value of the [illegible] between [illegible] Putting [illegible]

$$\frac{f(x+h)}{F(x+h)} = \frac{f(x+\theta h)}{F(x+\theta h)} \quad 0<\theta<1$$

Taking [illegible] and [illegible] we have the mean-value theorem.

19.

2. Before giving [illegible] proof I [illegible] to say a few words about each of the following proofs: Todhunter's, Mangeot's, Houel's.

(c) Todhunter's proofs are [illegible] the first and [illegible] proofs of [illegible] and are [illegible] from [illegible] the [illegible] of the differential [illegible] of the [illegible] assumed. In the [illegible] of each proof the generalised mean-value theorem is first proved.

(d) [illegible] proof [illegible] the same as the [illegible] proof given in the preceding article.

(e) [illegible] proof, although [illegible] the same as [illegible] proof given in Art. 18.

[illegible] theorem.

"We form the function

$$\psi(x) = f(x) - f(a) - \frac{x-a}{b-a}\{f(b) - f(a)\}$$

which vanishes for $x = a$ and $x = b$.

This [illegible] the whole of the interval [illegible] and is different [illegible] and determinate [illegible]

[illegible] and less than ϵ the differences

$$\psi(x' + \delta) - \psi(x'),\ \psi(x' - \delta) - \psi(x'),$$

when [illegible] the values of δ, and whence the ratios

$$\frac{\psi(x'+\delta) - \psi(x')}{\delta}, \quad \frac{\psi(x'-\delta) - \psi(x')}{-\delta}$$

when [illegible] zero shall have for all the values of δ opposite signs, so that their limits for $\delta = 0$, if [illegible] of opposite signs. But

[illegible] pp. [illegible]

[illegible]

$$\lim \frac{\psi(x' \pm \delta) - \psi(x')}{\pm \delta} = 0$$

is the only relation that can hold, and thus

$$f'(x') = \frac{f(b) - f(a)}{b - a}$$

[illegible]

$$b - a = h, \quad a = x, \quad b \text{ is } x + h,$$

and we have

$$f(x + h) = f(x) + hf'(x + \theta h), \quad 0 < \theta < 1.$$

§ 14.

[illegible] may be [illegible] that for two [illegible] that

$$\frac{f(b) - f(a)}{b - a} = f'(a + \theta\,\overline{b - a}), \quad 0 \angle \theta \angle 1;$$

which [illegible] point [illegible] graph [illegible] a point between [illegible] at that point [illegible] parallel [illegible] the extreme [illegible] the curve. The [illegible] therefore [illegible] in the [illegible] each of [illegible] that the [illegible] required are such that for [illegible]

[illegible]

[illegible] As regards the proofs [illegible] may be [illegible] and that excepting those of [illegible] which are [illegible] identical [illegible] the

other proofs have [illegible] *desired* [illegible] the first [illegible] of Cauchy [illegible] pointed out [illegible] of Art. [illegible] the same mistake [illegible] proof [illegible] strange that the same mistake [illegible] The proofs given [illegible] and Hermite [illegible] Cauchy's proof [illegible] The second proof [illegible] is open to the [illegible] that [illegible]

function has [illegible] in an interval [illegible]

in that interval. De Morgan's proof [illegible] that nowhere has [illegible] the form [illegible] being [illegible] defined [illegible] De Morgan [illegible] continuous with [illegible] his proof [illegible] open to no objection.

([illegible]) The [illegible] What is the necessary and sufficient condition [illegible] that [illegible] the relation

$$f(x+h)=f(x)+hf'(x+\theta h),\ 0<\theta<1,$$

holds? has not been answered as yet.

That for the validity of the above [illegible] the [illegible] the continuity of [illegible] is necessary [illegible] it is not necessary that $f'(x)$ be existent at every point [illegible] proved by the generalizations given by Prof. W. H. Young and [illegible] and by Dr. A. N. Singh.

Again, if at a point [illegible] and [illegible] then a sub-interval [illegible] h_2) can be found for which the relation will not hold, provided that [illegible] has [illegible] number of maxima and minima in the neighbourhood [illegible]

Some interesting results in connection with the question formulated in the beginning of this [illegible] have been given by Prof. Brouwer.[1]

[1] L. E. J. Brouwer, [illegible] *Amst. Akad.* [illegible] Vol. 17 [illegible] pp. [illegible] Vol. [illegible] pp. [illegible]

§ 16.

25. In [illegible] D. Pompeiu* gave a proof of the mean value theorem which differs from [illegible] other proofs [illegible] that it is not based on [illegible] [illegible] other proofs [illegible] proof established [illegible] the mean [illegible] deduced from [illegible] theorem. I proceed now to give this proof of Pompeiu.

Let a function [illegible] in an interval (the ends included) and [illegible] that interval [illegible] the given [illegible]. Then form the ratio

$$R(a,b)=\frac{f(a)-f(b)}{a-b}.$$

The question [is] to prove that there is at least one point between a and b the differential coefficient at which has precisely the value $R(a, b)$.

Now let

$$a_1=\tfrac{1}{2}(a+b).$$

Then we have

$$\frac{f(a)-f(b)}{a-b}=\frac{1}{2}\left\{\frac{f(a)-f(a_1)}{a-a_1}+\frac{f(a_1)-f(b)}{a_1-b}\right\}$$

or, briefly,

$$R(a,b)=\tfrac{1}{2}\{R(a,a_1)+R(a_1,b)\}. \qquad \ldots\ (1)$$

Two cases arise:—(1) the two ratios in the right side of the above equality are equal, in which case we may consider any one of them; (2) the two ratios are not equal so that one of them is greater, and the other less, than $R(a, b)$.

(2) Taking up the second case first, consider the function defined by

$$R(x,a_1)=\frac{f(x)-f(a_1)}{x-a_1}, \quad x\neq a_1$$

$$R(x,a_1)=f'(a_1), \quad x=a_1$$

Then obviously, for every value of x in the interval (a, b) other than a_1, $R(x, a_1)$ is continuous in x, [illegible] because of $R(a_1, a_1)$ being defined to be $f'(a_1)$ it is continuous at a_1. Thus $R(x, a_1)$ is continuous for every

* Sur le théorème des accroissements finis. Annales Scientifiques de l'Université de Jassy, 1905.

[illegible] of x in (a, b), the ends being [illegible] further because of the [illegible] points of [illegible] and [illegible] and [illegible] between [illegible] and [illegible] [illegible] must take the value [illegible] for some value b_1 of x intermediate between [illegible] and b, so that

$$R(b_1, a_1), [illegible] R(a_1, b_1), = R(a, b). \qquad \ldots \quad (2)$$

If the point b_1 coincides with a_1 then the theorem is proved as we will have

$$f(a_1) = R(a, b).$$

In any case

$$|a_1 - b_1| \leq \tfrac{1}{2} |a - b|.$$

If a_1 and b_1 are unequal, take

$$a_2 = \tfrac{1}{2}(a_1 + b_1)$$

Then proceeding as in the case of $R(a, b)$ it is proved that between a_1 and b_1 a point b_2 exists such that

$$R(a_2, b_2) = R(a_1, b_1) \qquad (3)$$

where

$$|a_2 - b_2| \leq \tfrac{1}{2} |a_1 - b_1|$$

We continue this reasoning indefinitely if none of the points b_k coincides with the corresponding point a_k.

We [illegible] that a [illegible] of intervals (I_k) possessing the following properties [illegible]

(1) The interval I_k, i.e. [illegible] a_k [illegible] b_k [illegible] inside the interval I_{k-1} and equal in length at [illegible] to one half of the length of I_{k-1} so that the length of I_k, i.e. $|a_k - b_k| \leq \frac{1}{2^k} |a - b|$

(2) $R(a_k, b_k) = R(a, b)$ for every value of the integer k [illegible].

Therefore I_k [illegible] point c as k is indefinitely increased, but when a_k and b_k tend to c from the definition of f it is obvious that $R(a_k, b_k)$ tends to $f(c)$

Thus it is proved that

$$\frac{f(a) - f(b)}{a - b} = f(c)$$

where

$$a < c < b$$

(β) If, in [illegible] $R(a, a_1)$ and $R(a_1, b)$ are equal then [illegible] end of [illegible] above, we have

$$R(a_1, b) = R(a, b)$$

so that $b_1 = b$ with

$$|a_1 - b_1| = \tfrac{1}{2}(a - b).$$

Proceeding as above [illegible] we get the same result in the end as there, viz. the mean value theorem is proved.

Criticism of Pompeiu's proof.

In order that Pompeiu's proof may hold it is further necessary, in addition to the existence of $f'(x)$, that [illegible] must be [illegible] and [illegible] have the same limit; [illegible] it does not necessarily follow that $\frac{f(x_n)-f(y_n)}{x_n-y_n}$ tends to $f'(\xi)$.

e.g. take [illegible] $f(x)=(x-\frac{1}{2})^2 \sin\frac{1}{x-\frac{1}{2}}$, $x_n=\frac{1}{2}+\frac{1}{2n\pi}$,

$$y_n=\frac{1}{2}+\frac{1}{2n\pi+\frac{\pi}{2}}.$$

Hence x_n, y_n both tend to $\frac{1}{2}$, but $\frac{f(x_n)-f(y_n)}{x_n-y_n}$ does not tend to $f'(\frac{1}{2})$, i.e., 0, but to $-\frac{2}{\pi}$.

§ 10.

We proceed now to consider the geometrical interpretation of the mean value theorem. If we assume that $f(x)$ admits of a graph for the whole interval (a, b), an assumption which involves departure from regularity only at a finite number of points, then the usual geometrical interpretation of the mean value theorem is this: To the chord joining any two points of the curve, $(x, f(x))$ and $(x+h, f(x+h))$, say PQ, there exists at least one tangent at an intermediate point p which is parallel to the chord. The truth of this interpretation is illustrated by the figures 1 and 2, the infinitude of $f'(x)$ at a point in the second figure not invalidating the theorem.

Fig. 1.

Fig. 2.

That the theorem may be untrue is illustrated by the figures 3–5.

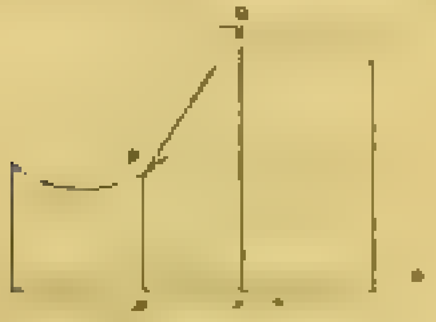

Fig. 3.

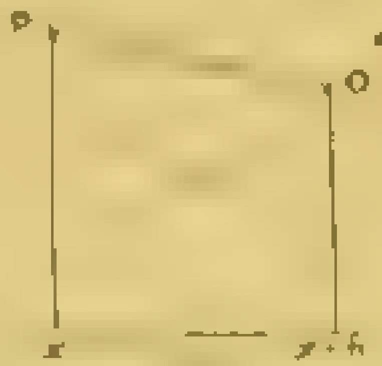

Fig. 4.

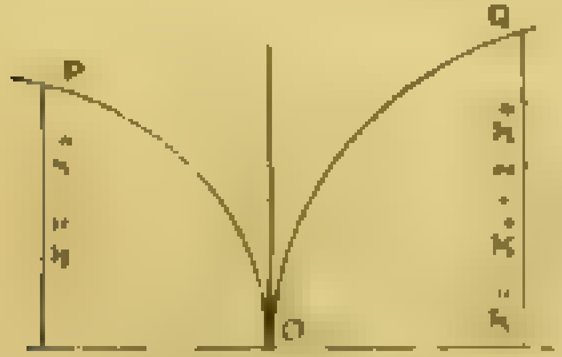

Fig. 5.

In Fig. 3 there is a discontinuity in $f(x)$ between P and Q; in Fig. 4 $f'(x)$ is non-existent at a point between P and Q; in Fig. 5 at O between P and Q there is a cusp but $f'(x)$ is non-existent there.

Other geometrical interpretations may be obtained by treating $f(x)$ as the area of the curve bounded by $y = f'(x)$ and the x axis, or by treating $f(x)$ as a volume.

§ 17

27. I will conclude this lecture by deducing a number of important theorems from the mean-value theorem in Dini's form, i.e., with the assumptions that $f(x)$ is finite and continuous in the interval (a, b), the ends being included, and that $f'(x)$ is existent at every point in the inside of the interval, the ends being excluded.

(i) *Theorem A.** If $f'(x)$ is zero at every point inside (a, b), then $f(x)$ is constant in the whole of the interval.

* Although it is evident that if $f(x)$ is constant in an interval $f'(x)$ is zero, and that $f'(x) > 0$ in an interval if at every point in that interval $f(x)$ is increasing, the converse propositions are not self-evident. The simplest logical proofs of these converse propositions are, as given here, by applying the mean value theorem.

Proof. Let x_1, x_2 be any two points in [illegible] then by the mean-value theorem

$$f(x_2)-f(x_1)=(x_2-x_1)\,f'(\xi) \qquad \dots \quad (1)$$

where [illegible] a point between [illegible] by the hypothesis. Therefore [illegible] thus the theorem is proved.

(c) Theorem [illegible] If [illegible] >0 at every point inside (a, b) then $f(x)$ is a constantly increasing function, [illegible] <0 at every point then $f(x)$ is a constantly decreasing function.

Proof. Applying [illegible] >0 and $x_2>x_1$ [illegible] >0. Thus $f(x_2)>f(x_1)$ [illegible] increases with [illegible] Similarly if [illegible] <0 at every point [illegible] that [illegible] constantly decreases with increasing x.

(d) Theorem [illegible] at any point of an interval [illegible] cannot be infinite.

Proof. Again applying (1),

$$f'(\xi)=\frac{f(x_2)-f(x_1)}{x_2-x_1}=\text{a finite quantity.}$$

Therefore inside (x_1, x_2) there is at least one point ξ where [illegible] is not infinite. Thus the theorem is proved.

(e) Theorem [illegible] is constant throughout an interval [illegible] at every point [illegible]

Proof. [illegible] applying [illegible] different from [illegible] and the theorem is proved.

(f) Theorem [illegible] If at a point [illegible] the limit of [illegible] exists then that [illegible] the value of [illegible] or [illegible] according as a is inside (a, b), is a or b.

Proof. Let a be an interior point then by the mean value theorem

$$\frac{f(a+h)-f(a)}{h}=f'(a+\theta h),\ 0<\theta<1.$$

Therefore as h tends to [illegible] $a+\theta h$ tends to a, consequently $f'(a+\theta h)$ tends to a definite limit say l by the hypothesis. But by the definition of the differential coefficient at a,

$$\lim_{h=0}\frac{f(a+h)-f(a)}{h}=f'(a).$$

Thus it is proved that [illegible] Similarly, the other cases can be dealt with.

Ibid

(f) *Theorem F.* If, in addition to the assumptions in [illegible] form of the mean value theorem, $f'(x)$ is continuous in (a, b), then corresponding to an arbitrarily small quantity $\delta > 0$ it is always possible to find a second quantity $\epsilon > 0$ and independent of x such that

$$\left|\frac{f(x+h)-f(x)}{h} - f'(x)\right| < \delta, \qquad \ldots \quad (2)$$

if $|h| < \epsilon$, whatever be the value of x in (a, b).

Proof. By the mean value theorem the left side of (2) is equal to $|f'(x+\theta h) - f'(x)|$. But by the hypothesis $f'(x)$ is continuous in (a, b), and therefore uniformly continuous. Therefore corresponding to δ a quantity ϵ exists such that

$$|f'(x_2) - f'(x_1)| < \delta,$$

whatever values x_2 and x_1 may have in (a, b) provided that $|x_2 - x_1| < \epsilon$.

Now $|\theta h| < |h|$; therefore

$$|f'(x+\theta h) - f'(x)| < \delta$$

if $|h| < \epsilon$.

(g) The result that $f'(x)$ takes every value between its upper and lower boundaries in (a, b), in addition to the assumptions in [illegible] form of the mean value theorem, it being assumed that $f'(a+0)$ and $f'(b-0)$ exist.[1]

Proof. By Theorem F, $f'(x)$ or $\phi(x)$ is either a continuous function of x or its discontinuities have been shown to be of the second kind only. Therefore, in any case, $\phi(x)$ takes every value between its upper and lower boundaries, as required by Weierstrass's theorem about continuous functions and by Darboux's theorem about functions which have discontinuities of the second kind.

An alternative proof[2] is the following.

Let U and L be the upper and lower boundaries of $f'(x)$ in (a, b). Then if $U > C > L$, values x_1 and x_2 of x exist such that in a neighbourhood of x_1, say $(x_1 - h_1, x_1 + h_1)$, the lower boundary of $f'(x)$ is L, and in a neighbourhood $(x_2 - h_2, x_2 + h_2)$ of x_2 the upper boundary is U, so that $f'(x_1)$ is $L + \epsilon_1$ and $f'(x_2)$ is $U - \epsilon_2$, ϵ_1 and ϵ_2 being quantities greater than 0 but as small as we please.

[1] Given by Darboux in Ann. de l'École Normale Sup., Series 2, Vol. 4, 1875, p. 109.

[2] See Dini's Calcolo Infinitesimale, [illegible] pp. [illegible]

Now consider the function

$$\phi(x) = f(x) - Cx$$

[illegible] obviously

$$\lim \frac{\phi(x_1 + h) - \phi(x_1)}{+h} < 0$$

$$\lim \frac{\phi(x_2 - h) - \phi(x_2)}{-h} > 0.$$

Hence there must [illegible] a number [illegible] such that for values of x between x_1 [illegible] and for values of x between $x_2 - \epsilon$ and x_2 [illegible] respectively the [illegible]

$$\phi(x) - \phi(x_1) < 0,$$

$$\phi(x) - \phi(x_2) < 0$$

[illegible] that [illegible] there must [illegible] where the [illegible] is attained [illegible] given number,

$$\phi(x_0 + h) - \phi(x_0) \geq 0,$$

$$\phi(x_0 - h) - \phi(x_0) \geq 0;$$

whence

$$\frac{\phi(x_0 + h) - \phi(x_0)}{h} \geq 0,$$

$$\frac{\phi(x_0 - h) - \phi(x_0)}{-h} \leq 0.$$

Therefore [illegible] at [illegible] the two expressions [illegible] tend to 0 with h [illegible] $\phi'(x_0) = 0$, i.e., $f'(x_0) = C$, which proves the theorem.

(b) *Theorem II.* [illegible] *Theorem* [illegible] then $f(x)$ cannot pass from [illegible] value B without taking all the intermediate values.

The proof [illegible] to the proof given above of Theorem [illegible]

THIRD LECTURE

GENERALIZATIONS OF THE MEAN VALUE THEOREM

§ 18.

23. To-day's lecture will deal chiefly with such results as establish the validity of the mean value theorem

$$f(x+h)=f(x)+hf'(x+\theta h),\ 0<\theta<1,$$

under conditions less restrictive than those in the usual form of the theorem; the mean value theorems, valid under less restrictive conditions may be considered generalizations of the theorem as found in the most up to date text-books on the Differential Calculus. To-day I will also give a number of deductions from these generalizations. I proceed now to give the generalization of W. H. Young and G. C. Young, which may be enunciated as follows:—

If, in a given interval (a, b), a function $f(x)$ is found to be finite and continuous, the end points being included, then for every pair of points (x_0, x_0+h) of (a, b), the ends being included,

$$f(x_0+h)=f(x_0)+hf'(x_0+\theta h),\ 0<\theta<1,$$

provided that at every point inside (x_0, x_0+h), the ends being excluded, there is no distinction of right and left with respect to the four derivates at that point, so that $D^+f(x)=D^-f(x)$ and $D_+f(x)=D_-f(x)$.

Proof:

As in the proof of the mean value theorem in Art. 21, let $\phi(x)$ denote

$$f(x)-f(x_0)-\frac{x-x_0}{h}\{f(x_0+h)-f(x_0)\}.$$

Then $\phi(x_0)$ and $\phi(x_0+h)$ both vanish. Also $\phi(x)$, like $f(x)$, is finite and continuous in (x_0, x_0+h) and does not show any distinction of right and left with regard to the derivates at any point inside (x_0, x_0+h), but at every such point

$$D^+\phi(x)=D^-\phi(x) \text{ and } D_+\phi(x)=D_-\phi(x).$$

Now [illegible] not [illegible] the [illegible] at most [illegible] upper and [illegible] lower bound different from zero [illegible] one of this upper or lower bound must be attained at an internal point of [illegible] Calling [illegible] this point there must exist [illegible] > 0 [illegible] sufficiently small [illegible] that taking h [illegible]

$$\left.\begin{array}{l}\phi(\xi+h)-\phi(\xi)<0\\ \phi(\xi-h)-\phi(\xi)<0\end{array}\right\}\text{ for } h<\epsilon$$

or

$$\left.\begin{array}{l}\phi(\xi+h)-\phi(\xi)>0\\ \text{[illegible]}\end{array}\right\}\text{ for } h<\epsilon.$$

In the first alternative [illegible] and [illegible] are both $\leq$ [illegible] and D [illegible] and D [illegible] ϕ [illegible] $\geq$ [illegible] In the [illegible] alternative the [illegible] are [illegible] [illegible] function of right and left

$$D \text{ [illegible]}$$

$$D_{+}\phi(\xi)=D_{-}\phi(\xi)=0,$$

and [illegible] therefore

$$f'(\xi)=\frac{f(x_0 \pm h)-f(x_0)}{h},$$

i.e., $f(x_0+h)=f(x_0)+hf'(x_0+\theta h),\ 0<\theta<1.$

§ 10

29. A few [illegible] the [illegible] generalization [illegible]

Ex. 1. Let [illegible] $\left(\text{[illegible]}\right)$ [illegible] $\leq$ [illegible] <1 [illegible]

[illegible] and [illegible] differential

[illegible] $\left\{\text{[illegible]}\right\}$ [illegible] the exception [illegible]

where, however, there [illegible] with respect to the [illegible][1] Although Dini's condition is not [illegible] the [illegible] holds and for every pair [illegible]

$$f(x_0+h)-f(x_0)=hf'(x_0+\theta h),\ 0\leq\theta\leq1$$

[illegible] $\left\{\text{[illegible]}\right\}$ [illegible] $\left(\text{[illegible]}\right)$

$$= h \text{ [illegible]},\ 0<\theta\leq1$$

[1] [illegible] differential coefficient [illegible]

[illegible] Let [illegible] $1 \leq x \leq 1$ then [illegible] finite [illegible] the interval [illegible] and has a differential coefficient [illegible] [illegible] the function [illegible] with regard to the [illegible] [illegible] the conditions [illegible] the mean value theorem holds.

§ 20

The next [illegible] to A. N. Singh:—

[illegible] finite and [illegible] required, [illegible] for every pair of points [illegible]

$$f(x_0+h)=f(x_0)+h f'(x_0+\theta h),\ 0<\theta<1,$$

provided that

(i) there [illegible] the progressive [illegible] differential [illegible] exists, while the other does not exist, and

[illegible] upper and lower derivatives [illegible] the other [illegible]

Proof.

Consider the function

$$\phi(x)=f(x)-f(x_0)-\frac{x-x_0}{h}\{f(x_0+h)-f(x_0)\}.$$

Then obviously [illegible] Then, unless $\phi(x)$ is zero throughout the interval [illegible] least one [illegible] or one [illegible] inside the interval.

Suppose [illegible] has a maximum at the point ξ within [illegible]. Then [illegible] the hypothesis [illegible] of the theorem, the derivates at ξ satisfy either the inequalities

$$\left.\begin{array}{l} D^+\phi(\xi) \geq D^-\phi(\xi) \\ D_+\phi(\xi) \leq D_-\phi(\xi) \end{array}\right\} A$$

or

$$\left.\begin{array}{l} D^+\phi(\xi) \leq D^-\phi(\xi) \\ D_+\phi(\xi) \geq D_-\phi(\xi) \end{array}\right\} B$$

Suppose that the inequalities (A) hold. Now as ξ is a point of maximum,

$$D^+\phi(\xi) \leq 0 \text{ and } D^-\phi(\xi) \geq 0. \qquad (a)$$

But by the inequalities (A),

$$D^+\phi(\xi) \geq D^-\phi(\xi).$$

Therefore [illegible] we have

$$D^+\phi(\xi) = D^-\phi(\xi) = 0. \qquad (b)$$

Again [illegible] for ξ is a point of maximum.

But

$$D_-\phi(\xi) \leq D^-\phi(\xi)$$

i.e., because of (b)

$$D_-\phi(\xi) \leq 0.$$

Therefore

$$D_-\phi(\xi) = 0 = D^-\phi(\xi)$$

Hence

[illegible] exists and is equal by [illegible] to [illegible]

Therefore [illegible] exists by the hypothesis [illegible] of the theorem and must be 0. Thus $\phi'(\xi)$ exists and is zero.

Hence

$$f'(\xi) = \frac{f(x_0+h) - f(x_0)}{h},$$

i.e. $f(x_0+h) = f(x_0) + hf'(x_0+\theta h)$, $0 < \theta < 1$.

In a similar manner, if the derivates at ξ satisfy the inequalities (B) it can be shown that

[illegible]

so that [illegible] and the theorem holds.

If ξ [illegible] point of minimum [illegible] shown similarly that the theorem holds.

Thus the theorem is completely established.

§ 21.

The following two examples illustrate Singh's generalisation [illegible]

Ex. 1. Let $f(x)$ equal $2x\sin\frac{1}{x}$ in the interval $(0, 1)$ and equal $x\sin\frac{1}{x}$ in the interval $(-1, 0)$. Then at the point $x=0$ inside $(-1, 1)$ there is a distinction of right and left with regard to the derivates, for

$$D^{+}f(0)=2,\ D_{+}f(0)=-2$$
$$D^{-}f(0)=1,\ D_{-}f(0)=-1,$$

so that

$$D^{+}f(0) \gtrless D^{-}f(0) \text{ and not equal; also}$$
$$D_{+}f(0) \text{ is unequal to } D_{-}f(0).$$

At all the other points the differential coefficient exists, and so there is no distinction of right and left. Also $f(x)$ is finite and continuous in $(-1, 1)$, the ends being included. Still for every pair of points (x_0, x_0+h) whatever, the theorem holds.

Ex. 2. Let $\psi(x)$ equal $2x$ [illegible] (x^2) in the interval $(0, 2)$, and be equal to x [illegible] x^2 in the interval $(-2, 0)$. Then for

$$f(x)=\sum_{n=1}^{\infty}\frac{\psi(x-\omega_n)}{2^n},$$

where $\{\omega_n\}$ is an enumerable and everywhere dense set of points in [illegible], the generalisation of Singh holds, so that the mean value theorem is valid although at every point of the set $\{\omega_n\}$ there is a distinction of right and left with regard to the derivates.

§ 22.

22. Singh has generalized the mean value theorem still further, by extending it to even certain types of discontinuous functions.[1] His second generalisation runs as follows

If inside a given interval (a, b) for which $f(x)$ is defined, there be a point X at which there is a discontinuity of the second kind at least on one side, say the right (left), and if inside a finite interval however small with the point of discontinuity X as left (right) end point

[1] In his paper [illegible] Hobson shows that the mean value theorem may hold for a discontinuous function like $\sin\frac{1}{x}$.

[illegible] other does not [illegible]

[illegible] other; then the mean-value theorem

$$f(x_0+h)=f(x_0)+h\,f'(x_0+\theta h),\quad 0<\theta<1,$$

holds.

Proof

If [illegible] then the proof of § 20 holds.

[illegible] which

$$\frac{f(x_0+h)-f(x_0)}{h}=f'(x_0+\theta h),\quad 0<\theta<1$$

[illegible]

$f(x) = -\phi(x, \beta)$, for $\beta - \gamma < x < \beta$.

It is easy to see [illegible] complementary [illegible] such an interval [illegible] In the first case [illegible] of Art. [illegible] those of the theorem of Art. [illegible] for any interval with [illegible] the points of the set G in $(x_0, x_0 + h)$.

§ 23

[illegible] interest as showing Dini's attempt at a general [illegible]

If a function f [illegible] and excepting [illegible] a_1, a_2 [illegible] uncertainty [illegible] points [illegible]

$$\frac{f(a_1 + \delta) - f(a_1)}{\delta} \quad [illegible]$$

with the [illegible] limit [illegible] two points [illegible]

$$f(x+h) - f(x) = h\,\lambda,$$

where λ is a quantity [illegible] never exceeds a given finite [illegible] positive number.

Let [illegible] at the most [illegible] then the theorem follows [illegible] theorem given in Art. 24 [illegible] of the points a_1, a_2 [illegible] $x + h$.

See [illegible] Dini.

[illegible] number [illegible] δ_1 and for $h = \pm \delta_1$ the ratios

$$\frac{f(a_1+h)-f(a_1)}{h},\quad \frac{f(a_2+h)-[illegible]}{h}$$

[illegible]

[illegible] e.g., the point a_1, in such a manner that we can put

$$x = a_1 \mp \delta,\quad x + h = a_1 \pm \delta'$$

[illegible] value

$$f(a_1 \pm \delta') - f(a_1) < \delta' A',\quad f(a_1 \mp \delta) - f(a_1) < \delta A',$$

and hence also in absolute value

$$f(a_1 \pm \delta') - f(a_1 \mp \delta) < (\delta + \delta') A',$$

and consequently

$$f(x+h) - f(x) = h h_1 A',$$

h_1 [illegible]

[illegible]

[illegible]

* Harnack [illegible] Thomae [illegible]

§ 24.

[illegible] the following two [illegible] are worthy [illegible] as generalizations of Rolle's [illegible]

[illegible] derivates[1]: If $f(x)$ is a function of x which is finite and [illegible]), the ends being included, and [illegible] a point ξ inside the interval [illegible] one of the upper derivates [illegible]

[illegible]

or [illegible] left and right

[illegible]

Proof

[illegible] the theorem is evident [illegible] has a positive upper [illegible] negative lower [illegible] and being [illegible] there is a point ξ [illegible] extreme value.

It then follows as in Art. 2[illegible] that

[illegible]

are respectively ≤ 0 and ≥ 0 [illegible]

[illegible] of ξ are [illegible] which proves the theorem.

[illegible] If [illegible] function defined in the interval [illegible] such that the upper and lower [illegible] then there [illegible] a point [illegible] of the two

[illegible]

[illegible]

[1] [illegible]

[illegible]

37. I shall conclude this [illegible]

from [illegible]

Theorem *

If y be a continuous function of x in the interval (x_0, x_1) and if the upper derivate to the right lies everywhere between the limits G and G' $(G > G')$, then the quotient

$$\frac{y'-y}{x'-x}$$

also lies between the same limits whatever arbitrary values be taken for x and x' in (x_0, x_1).

Proof.

Let $x' > x$. If $\frac{y'-y}{x'-x}$ were $> G$, then a quantity $\epsilon > 0$ could be chosen to be so small that, if z were defined by

$$z = y - (G+\epsilon)x,$$

the difference $z' - z$ would be > 0, say equal to δ. Therefore there must exist for z an upper bound x'' $(< x')$ where the equation

$$z' - z'' = \delta$$

would be satisfied last. At this point we should have

$$D^+ z'' \geqq 0,$$

and, consequently,

$$D^+ y'' \geqq G + \epsilon,$$

which would be against the hypothesis.

If, however, $\frac{y'-y}{x'-x}$ were $< G'$, then a quantity $\epsilon > 0$ could be chosen to be so small that, if z were defined by

$$z = y - (G'-\epsilon)x,$$

the difference $z' - z$ would be < 0, say equal to $-\delta$. Therefore there must exist for z an upper bound x'' $(< x')$ where the equation

$$z' - z'' = -\delta$$

would be satisfied last. At this point we should have

$$D^+ z'' \leqq 0$$

or

$$D^+ y'' \leqq G' - \epsilon,$$

which is again against the hypothesis.

If in the theorem instead of the upper derivate on the right we take any of the other derivates, the theorem will remain true and the proof will be similar to that given above.

* The enunciation as well as the proof is almost word for word the same as in Scheeffer's paper in *Acta Math.*, Vol. [illegible] ([illegible]), p. [illegible].

Theorem I. For a continuous function the four derivates all have the same upper bound and the same lower bound in a given interval and these upper and lower bounds are also the upper and lower bounds of the difference quotient

$$\frac{f(x_2)-f(x_1)}{x_2-x_1},$$

where x_1, x_2 take all possible values in the given interval.

Proof.

Let U and L denote respectively the upper bound and the lower bound of

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

Then it is obvious from Theorem I that

$$U \leq G, \quad L \leq G'.$$

We proceed to show that in the above the only signs permissible are those of equality.

If possible, suppose for example that $U < G$ and is equal, say, to $G - e$, where $e > 0$. Then it is not possible to find two values x_1, x_2 which will give

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} > G - e.$$

But this is absurd, because from the fact that the upper bound of $D^+ f(x)$ is G it follows that however small a positive number e may be taken, there are values x_1 and h such that

$$\frac{f(x_1+h)-f(x_1)}{h} > G - e.$$

Similarly it can be proved that L cannot be $< G'$. Thus it is proved that the upper derivate and the incrementary ratio have the same upper and lower bounds.

In the same manner, it can be proved that any of the other derivates has the same upper and lower bounds as the incrementary ratio.

Theorem II. If one of the derivates of a continuous function has a positive lower bound, then the function is monotone and increasing; if one of the derivates of a continuous function has a negative upper bound, then the function is monotone and decreasing.

This follows immediately from Theorem J.

Theorem [illegible] and $f(x)$ has a differential co-efficient at that point.

Proof:

Take any [illegible] interval [illegible] respectively the upper and lower [illegible] derivates [illegible]

[illegible] the [illegible] for all values of the variable.

Proof[1]:

Let [illegible] the upper derivate on the right [illegible]

$$\phi(x) \equiv cx + F(x) - f(x),$$

where $c > 0$ [illegible]

(a) It can be proved as follows that

[illegible]

If [illegible], arbitrarily small [illegible] for which

$$\frac{F(x+h) - F(x)}{h} > D^{+}F(x) - \delta$$

Also [illegible]

$$\frac{f(x+h) - f(x)}{h} < D^{+}f(x) + \delta.$$

Therefore it follows that arbitrarily small values of h exist for which

[illegible]

i.e.,
$$\frac{\phi(x+h) - \phi(x)}{h} > c - 2\delta$$

[1] The proof [illegible] Vol. 6, pp. 186-195.

[illegible] arbitrary,

$$D^{+}\phi(x) \geqq c$$

[illegible]

$$\phi(x'') - \phi(a) = 0,$$

and for every value of $h < x' - x''$ the relation

$$\phi(x'' + h) - \phi(x'') < 0$$

[illegible]

Hence [illegible] < 0 which is impossible.

[illegible] above, it follows that

[illegible]

[illegible]

[illegible] no value of x be < 0.

Hence it must be equal to 0.

[illegible] continuous function, then

[illegible] everywhere, and of potency c.

[illegible]

[illegible] are the same as those of [illegible]

[illegible]

$$L < K < U.$$

[illegible] within (a, b) such that

$$\frac{f(b_1)-f(a_1)}{[illegible]} = K.$$

[illegible] $f'(x_0)$ of $f(x)$ at x_0 equal to

$$\frac{f(b_1)-f(a_1)}{b_1-a_1}, \text{ i.e., } K.$$

[illegible] and of potency c.

[illegible] internal to the interval.

(γ)[1] Consider the incremental ratio

[illegible]

$$k_1 < k < k_2.$$

[illegible] of the [illegible] at which

$$m(x, y) = k_1$$

Similarly there is a point at which

$$m(x, y) = k_2.$$

[illegible] and P_1 are [illegible] P_2. [illegible]

[illegible]

[illegible] assumes the [illegible] at some intermediate point [illegible]

Theorem [illegible] and the differential [illegible] and lower limits.

Proof:

[illegible]

Theorem [illegible] and lower limits.

Proof:

The reasoning under [illegible] word [illegible] of the result [illegible] and [illegible] of page [illegible] therefore proved.

FOURTH LECTURE

The Functional Nature of θ

§ 25

[illegible] of θ —

Assuming (i) that $f'($ [illegible] in powers of h in the series

$$f'(x+\theta h) = \sum_{0}^{\infty} \frac{f^{(n+1)}(x)}{n!}\theta^n h^n$$

and $f(x+h)$ in the series

$$f(x+h) = \sum_{0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n,$$

[illegible]

$$\theta = \sum_{0}^{\infty} A_n h^n,$$

[illegible]

[illegible] and Pal's and similar to Whitcom's.

Since

[illegible]

[illegible]

[illegible]

Now [illegible] the coefficients of h [illegible] with the [illegible] given.

$$\frac{f^{(n)}(x)}{n!} = \sum_{K=1}^{n-1} \frac{f^{(K+1)}(x)}{(K+1)!}\, I_{n-1,K} \qquad n \geq [illegible] \qquad (2)$$

where $I_{n,K}$ denotes the coefficient of h^n in the expansion of

$$\left\{\sum_{r=0}^{\infty} A_r h^r\right\}^K.$$

From (2) we get the following equations first given by Whittaker[2] —

$$\frac{1}{2!} f^{(2)}(x) = A_0 f^{(1)}(x).$$

$$\frac{1}{3!} f^{(3)}(x) = A_1 f^{(2)}(x) + \frac{A_0^2}{2!} f^{(2)}(x)$$

$$\frac{1}{4!} f^{(4)}(x) = A_2 f^{(2)}(x) + \frac{2A_0 A_1}{2!} f^{(3)}(x) + \frac{A_0^3}{3!} f^{(4)}(x)$$

$$\frac{1}{5!} f^{(5)}(x) = A_3 f^{(2)}(x) + \frac{2A_0 A_2}{2!} f^{(3)}(x) + \frac{A_1^2}{2!} f^{(3)}(x) + \frac{3A_0^2 A_1}{3!} f^{(4)}(x)$$

$$+ \frac{A_0^4}{4!} f^{(5)}(x).$$

$$\frac{1}{6!} f^{(6)}(x) = A_4 f^{(2)}(x) + \frac{2A_0 A_3}{2!} f^{(3)}(x) + \frac{2A_1 A_2}{2!} f^{(3)}(x) + \frac{3A_0^2 A_2}{3!} f^{(4)}(x)$$

$$+ \frac{3A_0 A_1^2}{3!} f^{(4)}(x) + \frac{4A_0^3 A_1}{4!} f^{(5)}(x) + \frac{A_0^5}{5!} f^{(6)}(x)$$

$$\frac{1}{7!} f^{(7)}(x) = A_5 f^{(2)}(x) + \frac{2A_0 A_4}{2!} f^{(3)}(x) + \frac{2A_1 A_3}{2!} f^{(3)}(x) + \frac{A_2^2}{2!} f^{(3)}(x)$$

$$+ \frac{3A_0^2 A_3}{3!} f^{(4)}(x) + \frac{A_1^3}{3!} f^{(4)}(x) + \frac{6A_0 A_1 A_2}{3!} f^{(4)}(x)$$

* For large values of K the expression for $I_{n,K}$ is rather complicated. See recurrence formulae for the [illegible] paper [illegible] Vol 14, 1575, pp 79-64; also see [illegible] paper in [illegible] pp 19-16.

[2] [illegible] p 360

7

[illegible]

$$+\frac{6A_0{}^2A_5}{6!}f^{(6)}(x)+\frac{A_0{}^7}{7!}f^{(8)}(x).$$

Two more equations have been given by [illegible], those corresponding to

$$\frac{1}{9!}f^{(9)}(x) \quad \text{and} \quad \frac{1}{10!}f^{(10)}(x).$$

From the first equation we get [illegible] using the value of A [illegible] in the second we get A_1 and so on.

The [illegible] values of the first eight A's are

$$A_0 = \frac{1}{2}, \quad A_1 = \frac{1}{24}\,[\text{illegible}], \quad A_2 = \frac{1}{2}\left[\frac{1}{24}\,[\text{illegible}] - \frac{1}{24}\,[\text{illegible}]\right]$$

[illegible]

$A_3 = $ [illegible]

$A_4 = $ [illegible]

$A_5 = $ [illegible]

[illegible]

[illegible] The next problem [illegible] goes back, according to Professor R. [illegible] to the time of [illegible] and may be [illegible] as follows. Prove that [illegible]

[illegible]

[illegible]

as a function of h.

[illegible]

put

[illegible]

Then (M') gives

[illegible]

$$f(a+h) - f(a) - h f'(a) - \frac{h^2}{2} f''(a) \ [illegible]$$

[illegible]

[illegible]

Now [illegible] to zero the factor of h^2 [illegible]

$\frac{\epsilon}{2}$, where ϵ tends to 0 with h.

Thus (1) may be written

[illegible equation]

But by the mean-value theorem

$$f(x+h)=f(x)+hf'(x+\theta h)$$

Therefore, using (2) with the above,

[illegible equation]

[illegible equation] (3)

[illegible] where ϵ_1 tends to [illegible] with h. [illegible] the form

$$f''(x)+\epsilon_1=\frac{1}{2\theta}\left\{f''(x)+\epsilon\right\}$$

But [illegible] both the [illegible] of the above [illegible] we have

[illegible equation]

[illegible] enunciated in Art. [illegible] by making [illegible] continuity of [illegible] solution is as follows.

By the mean value theorem

[illegible equation] (1)

also by the mean value theorem [illegible]

[illegible equation]

therefore comparing (1) and (2) we have

[illegible equation]

i.e., putting θh for ξ

$$\theta f''(x+\xi_1) = \frac{f(x+h)-f(x)-hf'(x)}{h^2}. \tag{8}$$

[illegible]

[illegible]

where ϵ_2 tends to 0 with h.

Therefore (8) gives

$$\theta\{f''(x)+\epsilon_2\} = \tfrac{1}{2}\{f''(x)+\epsilon\},$$

from which because of $f''(x) \neq 0$, we have

$$\theta(+0) = \tfrac{1}{2}.$$

§ 27.

[illegible] Prove that

$$\theta'(+0) = \frac{1}{24}\,\frac{f'''(x)}{f''(x)}.$$

[illegible]

[illegible]

[illegible]

that [illegible]

[illegible] $0 < \theta < 1$

And

[illegible] (2)

[illegible] and $0 < \theta_1 < 1$.

Therefore comparing [illegible]

[illegible]

[illegible] 99,

[illegible] Hayashi, [illegible] the Tohoku Imperial University [illegible] Vol. XIII [illegible]

[illegible]

we have

$$\frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x + \theta_1 h) = h^2 \, [\text{illegible}] + \frac{h^3 \xi^2}{2} f'''(x + \theta_2 h) \qquad (8)$$

Now from § 20 [illegible], where [illegible] tends to 0 with h. Therefore (8) becomes, on dividing by h^2,

$$f''(x) + \frac{h}{6} f'''(x + \theta_1 h) = [\text{illegible}] f''(x) + \frac{h}{2} [\text{illegible}] f'''(x + \theta_2 h)$$

[illegible] $\frac{h}{6}$ [illegible] $\frac{h}{2}$ [illegible] $= \frac{h}{2}$ [illegible] (9)

putting

$f'''(x + \theta_1 h) = f'''(x) + \epsilon_1$, $f'''(x + \theta_2 h) = f'''(x) + \epsilon_2$, where ϵ_1, ϵ_2 tend to 0 with h.

Now $f''(x) \neq 0$, therefore dividing both the sides of (9) by $f''(x)$

$$\alpha = \frac{h}{24} \frac{f'''(x)}{f''(x)} + E$$

where $E = \frac{h}{f''(x)} \left\{ [\text{illegible}] \right\}$

Therefore E may be neglected in comparison with

$$\frac{h}{24} \frac{f'''(x)}{f''(x)}$$

as $f'''(x)$ is finite by hypothesis.

Thus it is proved that

$$\lim_{h \to 0} \frac{1}{h} [\text{illegible}] = \lim_{h \to 0} \frac{\theta - [\text{illegible}]}{h} [\text{illegible}] = \frac{1}{24} \frac{f'''(x)}{f''(x)}$$

42. If $f'''(x)$ exists and is finite, without any regard [illegible] the behaviour of $f'''(x)$ at points other than $t = x$, the above result holds, it being assumed that [illegible] the function $\theta(h)$ being single-valued.

[illegible] interval [illegible] The [illegible] with h

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots$$

$$+ \frac{h^n}{n!}\left\{ f^{(n)}(x) + \epsilon_n \right\}.$$

[illegible]

[illegible] the next lecture.

[illegible] taking for the sake of simplicity

$$x = 0,\ f(h) = \int_0^h \varphi(t)\,dt,\ \xi = \theta h$$

in the mean value theorem

$$f(x+h) = f(x) + h\, f'(x + \theta h)$$

Theorem [illegible] in the [illegible] between [illegible] and [illegible] h varying [illegible]

Theorem [illegible] the function [illegible] of any point [illegible] in the domain $\triangle$.

[illegible] 1844, p. XIX.

[illegible]

[illegible]

Cor. As a [illegible] function has an infinite number of maxima and minima in the neighbourhood of every point, it follows from the above theorem that [illegible] cannot be a nowhere differentiable function.

44. As in his first paper[1] [illegible] had postulated that θ should be not only *single-valued but also differentiable*,[2] Prasad pointed out in the following theorems that the single-valuedness of θ carried with it as a consequence its continuity but not its differentiability.

Theorem S. If θ is a single-valued function of h, then it is necessarily continuous everywhere with the possible exception of $h = 0$.

Theorem T. If $\theta(h)$ is single-valued and continuous, $\theta'(h)$ need not exist for every value of h.

45. Prasad gives certain general types[3] of $f(x)$ with the *necessary and sufficient conditions* for the existence of $\theta(+0)$ for each type. The types are the following:

Type I. $f(x) = \int (2 + \sin \psi(t))\, dt$ [illegible]

Here $\theta(+0)$ exists or does not exist according as

$$\psi = \log \frac{1}{t}$$

Or

$$\psi \neq \log \frac{1}{t}$$

Proof.

(*a*) Let $\psi = \log \frac{1}{t}$. Then [illegible] exists and equals 2, that is $f'(0)$ exists and is 2. Hence [illegible] condition is satisfied [illegible] the integrand being always positive, $f(t)$ is monotone. Thus θ is single-valued and $\theta(+0) = \frac{1}{2}$.

(*b*) Let $\psi \neq \log \frac{1}{t}$. Then it can be seen without difficulty that $f(t) = 2t + At \cos(\psi(t) + B) + o(t)$, where A is a constant differing from zero and B is another constant.

[1] See l. c. p. [illegible]

[2] See pp. 172-74 of his first paper. I take this opportunity to point out an error in that paper at the end of p. 173. The condition should be $\psi = \log \frac{1}{t}$ or $\psi \neq \log \frac{1}{t}$.

[3] See Prasad's paper "On the differentiability of the integral function [illegible]" *Crelle's Journal*, Bd. [illegible]; also Prasad's papers in the *Bulletin of the Cal. M. S.* Vol. [illegible]

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Also $f(h) = h^2 - A_1 h^2$ [illegible], A_1 being a constant $\neq 0$ and B_1 another constant.

Now by the mean value theorem *

$$f(h) = hw\,(\theta h).$$

Therefore

$\theta(+0)$ is non-existent as $\lim$ [illegible] is non-existent,

because if the limit $\theta(+0)$ existed

$$\lim \; [illegible] \quad \ldots \quad \ldots \quad (1)$$

would exist, which is not possible as $\theta \to +0$ [illegible] as is obvious from the form of the expression (1).

Type II. [illegible] $\int$ [illegible] (2 [illegible] λ [illegible] $\theta(+0)$ exists or does not exist according

[illegible]

or

$$\psi \leq \log \frac{1}{\theta} \, [illegible]$$

Proof.

(a) Let [illegible]. Then [illegible] $\int$ [illegible] X [illegible]

(a) $2X_1$ [illegible] (1)

(b) $2X_2(h)$ [illegible] (2)

where X_2 [illegible] denotes $\int X_1$ [illegible] dt.

But by the mean value theorem

$$f(h) = h\,[illegible]$$

* These are the correct conditions [illegible] of the first paper of [illegible]. Examples [illegible] and 3 obviously [illegible] the conditions given [illegible] Art. 14 and satisfy the correct conditions. The error seems to have crept in by some inadvertence.

Then [illegible], dividing both the sides of the above equation by $2X_2(h)$ and using (1) and (2), we have

$$1 \quad [illegible] \quad \frac{\sin \psi \, h \; [illegible]}{2X_2(h)} \quad [illegible] \quad \frac{h X_1(h\theta)}{X_2(h)} \quad \frac{h \, [illegible]}{2X_2(h)} [illegible]$$

$$= \cos \psi(h\theta) + \frac{h a_1(h)}{2X_2(h)}. \tag{3}$$

Now let h tend to 0 ; then (3) gives

$$1 = \operatorname{Lim} \left\{ \frac{hX_1(h\theta)}{X_2(h)} \right\};$$

the other terms in the equation being tending to 0. But X_2 is of the same order as hX_1; therefore the right side gives a function of θ say $\phi(\theta)$, whence we get $\theta(+0)$.

(b) The other case may be dealt with by using the methods of Arts. 10 [illegible] of Prasad's paper in [illegible] (1[illegible]), and it will be found that $\theta(+0)$ is non-existent.

46. *Illustrative Examples.*

Ex. 1. Let $f(x) = \int_0^x [illegible] \left\{ 2 + \sin \frac{1}{\sqrt{x}} \right\}$ [illegible]

Then $\theta(+0)$ exists and is equal to [illegible], although

$$f'(+0) = \operatorname{Lim}_{h \to +0} \frac{f(h)}{h} = \infty \text{ and } f'(h) = h^{[illegible]}\left(2 + \sin \frac{1}{\sqrt{h}} \right)$$

Ex. 2. Let $f(x) = \int_0^x x^{[illegible]} \left(2 + \sin \frac{1}{\sqrt{x}} \right) dx$

Then $f'(0)$ is ∞ and $\theta(+0)$ exists being [illegible]

Ex. 3. Let $f(x) = \int_0^x \left(2 + \sin \log^2 \frac{1}{x} \right) dx$

Then $\theta(+0)$ is non-existent.

§ 29

47. [illegible] with the [illegible] for each type [illegible] the types are the following

Types I and II. [illegible] Art. 15 that $\theta(+0) = \frac{1}{2}$. Now, as $\psi \to \log \frac{1}{u}$ [illegible]

[illegible]

But by the mean-value theorem [illegible]

Therefore

[illegible]

Dividing [illegible] the above by h^2

[illegible]

Now

[illegible]

Therefore $\theta(+0)$ [illegible] or not when [illegible]

Similarly as regards Type II it follows from Art. 45 that if [illegible] according as [illegible] $\to 1$ or not.

Type III. [illegible]

[illegible]

See page [illegible] of the [illegible] paper.

It has been proved by Prasad that $\theta'(+0)$ exists or not according as

$$\psi \succ \log \frac{1}{u}$$

or

$$\psi \preceq \log \frac{1}{u}$$

Type IV. $x(t) = \int_0^t W(u)\,du$, $W(u) = 2 + \int_0^u Y(v)\,dv$ [illegible]

$$Y(u) = \chi(u)\{2 + \sin \psi(u)\},\ \chi \succ 1,\ \psi \succ 1$$

$\theta'(+0)$ exists or not according as

$$\psi \succ \log \frac{1}{u}$$

or

$$\psi \preceq \log \frac{1}{u}$$

49. *Illustrative Examples*

Ex. 1. Let $x(t) = \int_0^t W(u)\,du$, $W(u) = 2 + \int_0^u Y(v)\,dv$ [illegible]

$$Y(u) = u^{-\frac{1}{2}} \left\{2 + \sin \frac{1}{\sqrt{u}}\right\}$$

Then $\theta'(+0)$ exists and is [illegible], although $f''(+0) = \infty$ and

$$f''(h) = h^{-\frac{1}{2}} \left\{2 + \sin \frac{1}{\sqrt{h}}\right\}.$$

Ex. 2. Let $x(t) = \int \left\{2 + \sin \frac{1}{u}\right\} du$

Then $\theta(+0)$ exists and equals $\frac{1}{2}$ but $\theta'(+0)$ is non-existent.

§ 30.

51. I will conclude to-day's lecture by giving, from Prasad, a [illegible] of functions θ each single-valued and each non-differentiable at the points of an everywhere dense set.[1]

[1] See pp. 179-80 of Prasad's first paper.

[illegible] Type 4.

Let $\phi(x)$ stand for [illegible] Then [illegible]

[illegible] increasing, and [illegible] for every value [illegible] positive or negative.

[illegible]

[illegible]

Therefore it is proved that for every value [illegible] > 0 and therefore $\phi'(x)$ is monotone and increasing, [illegible]

(b) The set $\{w_n\}$ be the set of rational points in the interval (0, 1).

$$f(h) = \sum_{n=1}^{\infty} [illegible]$$

will give [illegible] and [illegible] $= \sum_{n=1}^{\infty}$ [illegible] which will be monotone increasing and [illegible] for every value of h in (0, 1) because such is $\phi'(h - w_n)$.

Therefore by Theorem [illegible] there is a one-to-one correspondence between h and ξ, each moves in the same [illegible]. If [illegible] rational number [illegible] there [illegible] a corresponding value of h, say h_m. Now, by the mean-value theorem

[illegible]

But [illegible] exists for every value of h.

[illegible]

Thus it is proved that for $h = h_m$ [illegible]

$\frac{d}{dh}$ [illegible] exists and [illegible]

But $\left(\frac{d\xi}{dh}\right)_{h=h_m}$ must not exist, for if it were to exist $w'(\xi_m)$ would exist and [illegible] $\frac{\left[\frac{d}{dh}(w[illegible])\right]_{h=h_m}}{\left(\frac{d\xi}{dh}\right)_{h=h_m}}$, which will be absurd as $w'(\xi_m)$ is non-existent.

If ω is included in $\{\omega_n\}$ it can be proved that θ' [illegible] is non-existent because of the inclusion of the term $h^2 \sin \frac{1}{h^2}$ in $f(h)$.

Sec. 4. *For $f(h)$ a function taken from the Type II.*—The three non-differentiable functions given by Brodén* in Crelle's Journal, Bd. 118 are all monotone increasing and continuous. Therefore if $f(h) = \int_0^h u(t)\,dt$ where $u(t)$ is one of such functions there will be one to one correspondence between h and ξ as each moves in its domain of variability. Now each of the functions is non-differentiable at the points of an everywhere dense set, enumerable or unenumerable. Thus there are, corresponding to these functions of Brodén, functions θ each of which is non-differentiable at an everywhere dense set, enumerable or unenumerable.

* Denoting [illegible] the function by $f(x)$ for [illegible] where [illegible] primary points Brodén [illegible] the following [illegible] the three functions: (1) The function is continuous and [illegible] the derivatives $f'(x)$ and $f'_-(x)$ are [illegible] (and to $f'(x)$) with the exception of [illegible] and everywhere dense set of x-values [illegible] corresponding to the primary points. (See pp. 27 and 28 of Brodén's paper.) (2) [illegible] everywhere dense set of x-values (viz. those corresponding to the primary points) $f'(x) = 0$, and [illegible] $f'(x) = f'_+(x)$ [illegible] (3) [illegible] The first set [illegible] the primary points.

An interesting question arises. As in the case of each function of Bredon [illegible] is a primary point and therefore a point of non-differentiability for [illegible] is non-existent, what can be said about the existence of $\theta(+0)$ and $\theta'(+0)$?

The answer to this question may be attempted as follows:

(a) For each of the first two functions [illegible] which has been designated $f''(\cdot)$ in the preceding pages, exists, is finite and greater than 0; hence in the case of each [illegible] $\theta(+0)$ exists and equals $\frac{1}{2}$. In the case of the third function of Bredon [illegible] $= \infty$, and it is difficult to prove that $\theta(+0)$ exists.

(b) As regards $\theta'(+0)$, its existence is unlikely, for in any neighbourhood of 0, however small, the points where θ is non-differentiable are everywhere dense.

FIFTH LECTURE

The Functional Nature of θ (Continued).

§ 31

51. Just as the [illegible] lecture [illegible] the [illegible] functional nature of θ [illegible] function [illegible] will [illegible] discussed chiefly the functional nature [illegible] function of h. Before [illegible] research [illegible] θ as a multiple-valued function I reproduce [illegible] interest the following passage from the first important [illegible] the [illegible] paper[1] [illegible] on the law of the mean":—

(a) "The formula

$$[illegible] \; (x+h) - [illegible] = [illegible] \quad [illegible] < \theta <$$

is known to every student [illegible] the law of the mean or [illegible] synonym. It is [illegible] the formula [illegible] is defined in an interval [illegible] $\leq x \leq$ [illegible] and [illegible] derivative [illegible] exists in the continuum $a < x < b$.

The quantity ξ which [illegible] is [illegible] function of the two independent variables x and h and is [illegible] values of [illegible] and h for which [illegible] interval in which [illegible] exists. The purpose of the present paper is to discuss the [illegible] quantity ξ, and we shall write

$$\xi = \xi\,(x, h)$$

whenever [illegible] to emphasize [illegible] functional character.

In studying the function ξ [illegible] $\xi(x, h)$ is continuous [illegible] continuous [illegible] A [illegible] inspection would tend [illegible] that [illegible]

[1] Annals of Mathematics [illegible]

[illegible] the Mean [illegible] by the writer of the [illegible] Annals [illegible] Law of the Mean [illegible] by the writer [illegible] to p. 177 of the Annals).

[illegible] everywhere (see p. [illegible]).

[illegible] fully understood.

[illegible] tenuous cases.

[illegible] that value is numerically less than h.

[illegible]

$$y=f(x)=x^2\left(1+\sin\frac{1}{x}\right) \text{ if } x\neq 0.$$

[illegible]

The derivative is

[illegible]

$$y' = f'(x) = 0 \qquad \text{if } x = 0,$$

[illegible]

For it is obvious[1] that

$$f'(x) = 0 \text{ when } x = \frac{2}{(4n+1)\pi}$$

and that

$$[illegible] \frac{2}{[illegible]}, \quad f'(x) = 0 \text{ when } x = \frac{2}{[illegible]}$$

[illegible] the r[illegible] [illegible] $<$ [illegible]

of x [illegible] $\frac{2}{[illegible]}$ and $\frac{2}{[illegible]}$ [illegible]

[illegible]

$$f(h) - f(0) = h\, f'(\xi) \quad 0 < \xi < h,$$

the right-hand side is never negative.

We may therefore conclude that

[illegible] continu[illegible] assumes equal values any, for all values [illegible] neighbourhood of any point where [illegible]

§ 82

[illegible] the [illegible]

[illegible]

[illegible]

Ex. [illegible]

(ii) *Theorem:* The roots of the equation in θ, [illegible] form either a finite [illegible] or [illegible] enumerable and nowhere dense set.

Proof.

Let [illegible] Then [illegible] according to a known theorem [illegible] the [illegible] maxima and minima of every continuous function form a finite or enumerable set, the values of t for which

$$F'(t) \equiv f'(t) - \frac{f(h)}{h} = 0$$

form such a set [illegible] form such a set.

That [illegible] the set [illegible] everywhere dense is obvious.

For if the set were everywhere dense [illegible] could be [illegible] upon as a limiting point [illegible] and would therefore itself satisfy [illegible]. Therefore [illegible] number [illegible] which we [illegible] barred.

[illegible] the values of θ satisfying [illegible] may be arranged in order of [illegible] and denoted by [illegible], being the greatest [illegible]. Pasch calls θ_1 [illegible] the principal value of θ for a given h [illegible] θ_1 [illegible] single-valued function of h.

§ 33.

5. The following simple examples illustrate the theorem and definition of § 32.

Example 1. [illegible]

$$[illegible]$$

Then [illegible] equation

$$[illegible]$$

Therefore [illegible] value [illegible] becomes

$$[illegible]$$

[1] See p. 1 [illegible] Leipzig [illegible] Buchhandlung [illegible]

Therefore θ is given by the [illegible] equation

$$\cos\left(\frac{1}{2}\log\frac{1}{\theta^2h^2}\right) = [illegible] \cos\left(\frac{1}{2}\log\frac{1}{h^2} + \frac{\pi}{4}\right)$$

Taking α to be the numerical [illegible] whose cosine equals the right side of the above equation, and [illegible] $N(h)$ a suitable integer dependent on h, we have

$$\frac{1}{2}\log\frac{1}{\theta^2h^2} = 2N(h)\pi \pm \alpha,$$

i.e., taking h to be positive,

$$\theta = \frac{1}{h}e^{-2N\pi \mp \alpha},$$

where the integer N is such that θ lies between 0 and 1.

If N_1 is any such integer for a given h, it is [illegible] that N_1+1, N_1+2, [illegible] are all such integers. For instance if h is $e^{[illegible]}$, [illegible] being an integer,

[illegible]

thus

[illegible]

and

$\theta = [illegible]$

The principal value θ_1 is $e^{-\frac{\pi}{2}}$, the other values in order of magnitude are

$$e^{[illegible]}, \quad e^{[illegible]}, \quad e^{[illegible]}, \quad e^{[illegible]}, \quad \ldots$$

Example II. Take

$$f(x) = \int_0^x \cos\frac{1}{t}\,dt.$$

Then $f'(x)$ exists for every value of x including 0.

Therefore the mean value theorem (I) of [illegible] becomes

$$f(h) = h\,f'(\theta h) = h\cos\frac{1}{\theta h}.$$

Therefore θ is given by the transcendental equation [?]

$$\cos\frac{1}{\theta h}=\frac{f(\theta h)}{h},$$

in which the right side [illegible] $\frac{1}{h}$ [illegible] values of h. Taking

[illegible] the numerically least angle [illegible] the right side of the above equation, we have

$$\frac{1}{\theta h}=2N(h)\pi\pm\alpha,$$

i.e., $$\theta=\frac{1}{h(2N\pi\pm\alpha)},$$

where N is any integer so chosen that θ lies between 0 and 1.

If N_1 is such an integer, then obviously N_1+1, N_1+2, [illegible] are all such integers.

For instance, if $h=\frac{1}{2m\pi+\frac{\pi}{2}}$, [illegible] any large integer

[illegible]

thus [illegible]

and [illegible] nearly

The principal value θ_1 is [illegible] nearly; the other values

[illegible]

in order of magnitude, are nearly

$$2m\pi + \frac{\pi}{2}, \qquad 2m\pi + \frac{\pi}{2},$$

$$2(m+1)\pi - \left(\frac{\pi}{2} + \frac{1}{2m\pi + \frac{\pi}{2}}\right), \qquad 2(m+1)\pi + \frac{\pi}{2} + \frac{1}{2m\pi + \frac{\pi}{2}},$$

[illegible]

§ 34

34. That the principal value of θ is not necessarily continuous at $h = 0$ is proved by the consideration of $\theta_1(h)$ in the case of Example I of § 33.

Proof. Let h tend to 0 by taking the values of the form $e^{-2m\pi}$ where m is positive and integral; then as shown in connection with the study of Example I in § 33, $\theta_1(h)$ is always $e^{-\frac{\pi}{2}}$. Thus for this mode of approach $h \to 0$, $\theta_1(h)$ tends to $e^{-\frac{\pi}{2}}$.

If h is of the form $e^{-\left(2m\pi + \frac{\pi}{2}\right)}$, where m is positive and integral, then

$$\frac{1}{\sqrt{2}} \cos\left(\frac{1}{2} \log \frac{1}{h} + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\cos [illegible] \frac{\pi}{2} + \frac{\pi}{4}\right) [illegible]$$

Therefore [illegible] and θ [illegible] is equal to [illegible]

Thus for this mode of approach to 0 for h, [illegible] by taking values of the form

$$[illegible]$$

$\theta_1(h)$ tends to [illegible]

Therefore $\theta_1(h)$ [illegible]

§ 35.

35. That the principal value [illegible] not [illegible] at $h =$ [illegible] not necessarily differentiable there is proved [illegible] Example [illegible]

In this case in the [illegible] of $h =$ [illegible] behaves as

$$-h^2 \sin\frac{1}{h}$$

Therefore, as shown in § 33,

$$\theta(h) = \frac{1}{h\,(2N\pi \pm \alpha)}$$

and $\theta_1(h)$ tends to 1 as h tends to 0 [illegible] that $\theta_1(h)$ is [illegible] proved [illegible] at $h = 0$.

36. I proceed now to prove that $\theta_1(h)$ is non-constant.

Proof:

(a) Consider

$$\theta_1(h) - 1$$

for the values of h of the sequence

[illegible]

Then we have [illegible] equation

[illegible]

and the form [illegible]

[illegible] higher powers of $\frac{1}{n}$ than the first

[illegible]

Therefore [illegible]

and

[illegible] higher powers of $\frac{1}{n\pi}$ than the first [illegible]

[1] [illegible]

Hence for the sequence under consideration [illegible]

$$\operatorname*{Lim}_{h=+0} \frac{\theta_1(h)-1}{h}=0.$$

(b) Again consider

$$\frac{\theta_1(h)-1}{h} \text{ [illegible]}$$

for the values of the sequence

$$\left\{\frac{1}{(2m+1)\pi}\right\}$$

Then we have x of the form

[illegible] + higher powers of [illegible] than the first.

Therefore

$$\theta_1(h) \text{ [illegible]}$$

and

$$\frac{\theta_1(h)-1}{h}=(2m+1)\pi \left\{ \frac{\text{[illegible]}}{(2m+1)\pi+2} \text{ [illegible]} + \text{higher powers of [illegible] than the first} \right\}.$$

Hence for the sequence [illegible]

$$\operatorname*{Lim} \text{ [illegible]}$$

which is different from the limit [illegible]

Therefore

$$\operatorname*{Lim}_{h=+0} \frac{\theta_1(h)-1}{h}$$

is non-existent and consequently $\theta_1(h)$ has no differential coefficient at $h=0$.

§ 30.

I [illegible] statement [illegible] passage [illegible] from H[illegible] paper that even [illegible] [illegible] The [illegible] has [illegible] to [illegible] not distinguishing between the different values of ξ for a given h.

[illegible] then the corresponding [illegible] Art. [illegible] of my first paper [1] is quite applicable to ξ_1.

[illegible] for any value of h other than 0.

[illegible] statement [illegible] examine his example

$$f(t) = t^2 \left(1 + [illegible]\right) [illegible]$$

$$f(0) = 0$$

[illegible] point [illegible] of the single valued func[illegible] h. The [illegible] point of discontinuity of the first kind [illegible] sequence $\{h\}$ tending [illegible] the correspond[illegible] $\{ \}$ [illegible] tend [illegible] ξ_1 [illegible] from ξ_1. Then

$$\frac{f(h)}{h} = f'(\xi_1), \quad \frac{f(h)}{h} = f'(\bar{\xi}_1),$$

[illegible] there are two different values [illegible] of [illegible]. Therefore ξ_1 [illegible] which is absurd.

Again [illegible] a point of discontinuity of the second kind [illegible] there must be a sequence $\{h\}$ tending to h, the sequence [illegible] corresponding [illegible] which does not tend to any limit. Therefore [illegible] a neighbourhood of h as small as we please there must be values of ξ_1 [illegible] ξ_1 [illegible] differing from one another by more than any [illegible] chosen quantity $e > 0$. But [illegible] are different from one another by a quantity as small as we please [illegible] of the

[1] See the second foot-note on p. [illegible]

continuity of $\frac{f(h)}{h}$ at h. Therefore

[illegible]

[illegible]

differ from one another [illegible] possible [illegible]

§ 37

[illegible] I proceed [illegible] study of the interesting case [illegible] function [illegible] is nowhere differentiable. Take

$$f(x) = \int \omega(x)\, dx$$

where [illegible] is a continuous but nowhere differentiable function, say Weierstrass's function

$$= \sum_{1}^{\infty} \cos ([illegible])$$

Then the [illegible] nowhere differentiable function.[*]

Proof

(i) that [illegible] every value of $h > 0$ [illegible]

$$\frac{f(h)}{h} = \omega(\theta_1 h).$$

[illegible]

(ii) that ω being a continuous function [illegible] argument [illegible] argument $\theta_1 h$ and consequently θ_1 must be continuous.

(b) The non-differentiability of [illegible] the equation

[*] See Prasad's quoted paper.

[illegible] $= h$ [illegible]

where ξ [illegible] the right [illegible] differentiate [illegible] for every value of h. Therefore

$$f'(h) = \omega(\xi) + h\frac{d}{dh}\{\omega(\xi)\}.$$

[illegible]

[illegible]

But [illegible] would exist [illegible] being equal [illegible]

[illegible] Therefore [illegible] non-existent [illegible] proved that ξ and, consequently [illegible]

§ 34

(a) The next step [illegible] study [illegible] differentiate θ_1 is to [illegible] corresponding to any given value of h [illegible] has actually given a formula [illegible] for the value $h =$ [illegible] the formula [illegible] Whatever [illegible] may be, there is a value of θ between

$$\frac{1}{2} + \frac{1}{13^{\frac{1}{2}}} \text{ and } \frac{1}{2} + \frac{1}{13^{\frac{1}{2}}},$$

and another value between

$$\frac{1}{2} - \frac{1}{13^{\frac{1}{2}}} \text{ and } \frac{1}{2} - \frac{1}{13^{\frac{1}{2}}}.$$

* See [illegible] paper [illegible] which the [illegible] carefully treated. The [illegible] of θ [illegible] [illegible] Therefore θ is given [illegible] or [illegible] [illegible] zeroes of the function [illegible]

† In [illegible] paper [illegible] of Weierstrass [illegible] [illegible] of that paper, for "the zeroes" read "zeros."

so that the value of θ [illegible] and [illegible] are given by the expression

$$\frac{1}{2} + \frac{1}{2}\left(1 \quad [illegible] \right)$$

λ_1 being [illegible] or between [illegible] and 1 which can be [illegible] and [illegible] between [illegible] and [illegible] and [illegible] the expression [illegible] where [illegible] is a [illegible] number between [illegible] and [illegible] or [illegible] and [illegible].

(d) In order to answer the question [illegible] as h [illegible] when [illegible], N being an integer. (e) It is obvious that the value of u in this case is [illegible] the value of u_n [illegible] the [illegible]. Thus as h decreases from [illegible] each θ decreases [illegible] consider the behaviour of u [illegible] as h [illegible] from [illegible] successively $h =$ [illegible], $h =$ [illegible] [illegible] in the former case the [illegible] of u [illegible] and [illegible] and in the latter case [illegible] and [illegible]. We may therefore easily conclude that as h [illegible] tends to [illegible]

§ 30.

(2) The next question to be considered [illegible] that [illegible] is non-existent for $h > 0$ does [illegible] exist or [illegible] to be 1?

This is a question of great difficulty and can be only briefly considered here. For a fairly detailed consideration see Pringsheim's [illegible] paper.

There are non-differentiable functions which have such a progressive differential coefficient at $h = 0$; there are others for which this is not true. So there are all sorts of possibilities for the behaviour of $\frac{u_n(h) - 1}{h}$ as h tends to 0.

§ 40.

[illegible] the following question [illegible] true that [illegible] every prescribed function [illegible] in case [illegible] is single valued or [illegible] prescribed set [illegible] multiple [illegible] the mean value theorem

$$viz., \quad f(h) = hf'(h\theta)$$

[illegible]

$$f(h) = hf'(h\theta_n)$$

[illegible] what conditions must be satisfied by the [illegible] or the prescribed set $\{\theta_n(h)\}$ in order that $f(h)$ may exist?

[illegible] consider the [illegible] as a single valued function.

(1) If we [illegible] expansible in the form $\theta(h) = a_0 + a_1 h + a_2 h^2 + \ldots$ [illegible] search for [illegible] expansible in the form

$$f(h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \ldots \text{ to infinity},$$

[illegible] and A_1 [illegible] we have by the mean value theorem

$$f(h) = hf'(\theta h),$$

[illegible]

[illegible] we get [illegible] values of the A's [illegible] the [illegible]

[illegible] but $A \neq 0$ [illegible] must be $\left(\frac{1}{3}\right)^{\frac{1}{2}}$, generally, if

$$A_2 = A_3 = \ldots = A_{m-1} = 0 \text{ but } A_m \neq 0,$$

then [illegible] be $\left(\frac{1}{m}\right)^{[illegible]}$. [illegible] condition the A's are indeterminate and expressible in terms of [illegible] if $A_2 \neq 0$ [illegible]

[illegible]

[illegible] single valued function [illegible] that [illegible] everywhere non-differentiable [illegible]

It is thus clear from [illegible] and [illegible], that [illegible] but [illegible] may exist corresponding to $\theta($ $)$, [illegible] being [illegible] 0 and 1 for [illegible] values of h, θ must fulfil certain restrictions [illegible] arbitrary. The following examples will illustrate this remark.

Ex. 1. If θ is [illegible] constant, it can take values only of the form $\left(\frac{1}{n}\right)^{n-1}$ where n [illegible].

Ex. 2. There is [illegible] corresponding [illegible] $\frac{1}{2}$ [illegible] $f(h) = A_2 \{$ [illegible] $\}$, A_2 being any constant different from 0.

$$\text{For } \theta = \frac{1}{\sqrt{3}} \ldots, \; f(h) = A_3\left\{h^3 + \frac{6}{8\sqrt{3}-4}h^4 + \ldots\right\}$$

A_3 being any constant different from 0.

Ex. 5. If $\theta = \frac{1}{2}$ [illegible] $\frac{2}{}$ [illegible] for each value $f(h) = h^2 - h^4 \sin\frac{1}{h}$.

Ex. 6. If $\theta^{\frac{1}{2}} = \frac{2}{3} - 8h^{\frac{1}{2}}\left(\frac{2}{3}\right)$ [illegible] for small values of h, then for such values [illegible].

Ex. 7. If

$$\theta = [\,\text{[illegible]}\,]$$

then $f(h)$ will be of the form $Ch^2 + \ldots \cos\left\{\log\frac{1}{h} + D\right\}$, C and D being suitable constants, and in fact C = [illegible], $D = \tan^{-1}\frac{1}{2}$.

61. Let us consider now the case in which $\theta(h)$ is multiple-valued.

[illegible] example [illegible]

[illegible] and is for values of h in $(\frac{1}{2}, 1)$ equal to

$$\frac{h-\frac{1}{2}h^2-\frac{1}{8}}{h} \text{ or } \frac{\frac{1}{2}h^2+[illegible]}{h}$$

[illegible] on the assumption that it is a power series in h.

[illegible] that $w(t)$ is continuous.

[illegible] of ξ being $\frac{h}{2}$.

[illegible]

$$w(t) = a_0 + a_1 t + a_2 t^2 + \dots; \frac{1}{2} \leq t \leq 1.$$

Then [illegible]

$$\xi = [illegible] \qquad (I)$$

Also,

$$[illegible] \qquad (II)$$

according as [illegible]

From the first alternative [illegible], we have

$$\left(a_0 h - \frac{a_1}{2} h^2 + \dots \right) [illegible]$$

Hence $a_0 = 1$, $a_1 = -1$, $a_2 = [illegible] = \dots = 0$.

Then values of the [illegible] satisfy the second [illegible] formal [illegible] of (11) [illegible] and [illegible] (1) [illegible] (1, ½) [illegible] ½). And

$f(h)=\frac{h^2}{2}$ or $h-\frac{h^2}{2}-\frac{1}{4}$ according as h is in $(0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$.

Ex. 2. If ξ is given by the following scheme, find $f(h)$ on the same assumption as in *Ex.* 1.

Dividing the interval (0, 1) into four quarters $\left(0, \frac{1}{2^2}\right)$, $\left(\frac{1}{2^2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{3}{2^2}\right)$, $\left(\frac{3}{2^2}, 1\right)$ and calling them respectively the first, second, third and fourth quarters, ξ is equal to $\frac{h}{2}$ when h is in the first quarter, for values of h in the second quarter ξ or $\frac{1}{2}-\xi$ equals

$$\frac{-\frac{h^2}{2}+\frac{1}{2}h-\frac{1}{2^4}}{h},$$

according as ξ is in the first or second quarter; for values of h in the third quarter ξ, $\frac{1}{2}-\xi$ or $\xi-\frac{1}{2}$ equals

$$\frac{\frac{h^2}{2}-\frac{1}{2}h+\frac{3}{2^4}}{h}$$

according as ξ is in the first, second or third quarter; and lastly for values of h in the fourth quarter ξ, $\frac{1}{2}-\xi$, $\xi-\frac{1}{2}$ or $1-\xi$ equals

$$\frac{\frac{h^2}{2}-h+\frac{3}{8}}{h}$$

according as ξ is in the first, second, third or fourth quarter.

(c) For the case in which $\theta(h)$ has a finite number of values see Prasad's paper "On the determination of $f(x)$ corresponding to a given Rolle's function $\theta(h)$ when it is multiple valued" (*Proc. Benares M. S.*, Vol. XII).

SIXTH LECTURE

[illegible] MEAN VALUE THEOREMS. GENERALIZED FUNCTIONS θ

§ 41

[illegible] To-day I will first [illegible] the [illegible] function θ as a function of [illegible] and h, and then proceed [illegible] a number of [illegible] theorems and results, each of which [illegible] connected with the mean value theorem [illegible] function θ as a [illegible] generalization [illegible] θ [illegible] I proceed to formulate and prove in [illegible] the following three theorems.

Theorem I. [illegible] x and h [illegible] real [illegible] $\leq b$ and also $a \leq x+h \leq b$; further let $f(x)$ be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. Then of all the functions $f(x)$ satisfying the above conditions [illegible] has the property to satisfy the mean value theorem

$$f(x+h)-f(x)=hf'(x+\theta h), \tag{M_1}$$

with a value [illegible] $\theta =$ [illegible]

Proof.

[illegible] on the [illegible] side of [illegible] then is the left side and therefore [illegible] the right side of M_1 differentiable with respect to [illegible] consequently is [illegible] because of [illegible] twice differentiable. By differentiating both sides of (M_1) with respect to x and with respect to h we have

$$(1)\qquad \begin{cases} f'(x+h)-f'(x)=h\,f''(x+\theta h), \\ f'(x+h) \; [illegible] \end{cases}$$

From these two equations follows immediately

$$[illegible]$$

Differentiating the above [illegible] with respect to h, we have

$$(2)\qquad [illegible]$$

Now $f''(x)$ is differentiable [illegible] the inside [illegible] and therefore continuous, and consequently [illegible] there. Consequently from (2) we have for $h=0$

$$[illegible] (1-2\theta) = [illegible]$$

If now $\varphi'(x)$ were identically equal to 0 then $f(x)$ would be [illegible] and the formula (M_1) would be satisfied for every arbitrary value of θ. This trivial case shall be here, as also in the following pages, excluded.*

Therefore θ must be $\frac{1}{2}$.

Thus from (2) we have

$$f''(x+h)=f''\left(x+\frac{h}{2}\right).$$

Since h is permitted to take arbitrary values for which $x+\frac{h}{2}$ and $x+h$ are points inside of (a, b) therefore it follows that $f''(x)$ must be a constant different from 0 and therefore $f(x)=\alpha x^2+\beta x+\gamma$, α, β, γ being constants.

(C) *Theorem II.* If in the mean value theorem (M_1) $\theta=\theta(h)$ [illegible] and further $\theta(0)$ [illegible], then of all the functions $f(x)$ which are continuous in (a, b), the ends being included, and differentiable in (a, b), the ends being excluded, [illegible] alone has the property to satisfy (M_1) with such θ as stated above, which [illegible]. This $\theta(h)$ is $\frac{1}{\alpha h}\log$ [illegible]

Proof.

First it follows from (M_1) by the same considerations as before that [illegible] and therefore also [illegible] for [illegible] with respect to x and with respect to h we have from (M_1), putting ξ for θh

$$(3)\quad \begin{cases} f(x+h) = f(x)+hf'(x+\xi) \\ f'(x+h) = f'(x+\xi)+hf''(x+\xi)\xi', \end{cases}$$

where $\xi'=\frac{d\xi}{dh}$. One can assume that ξ' is not identically zero, for otherwise would ξ be constant and from the above equation would also $f''(x)$ be a constant which would lead to the trivial case of $f(x)$ as a linear function. Therefore it follows from the two equations by the elimination of [illegible] as is easily seen, that

$$(4)\quad f'(x+h)(1-\xi')+\xi' f'(x)=f'(x+\xi)$$

or

$$\xi'\{f'(x+h)-f'(x)\}=f'(x+h)-f'(x+\xi).$$

Now however this equation can be always differentiated with respect to [illegible]

* If $f'(x+h)=f'(x)$ [illegible] for arbitrary values of h and x then must $f'(x)$ be constant in the whole region [illegible] as the case [illegible]. However, if for a definite value of h and arbitrary values of x $f'(x+h)=f'(x)$ [illegible]

Consequently (I) admits of a solution with respect to ξ' [illegible]

$$\xi' = \frac{f'(x+h) - f'(x+\xi)}{f'(x+h) - f'(x)}$$

The right side [illegible] independent of x, is differentiable with respect to x. Therefore [illegible] exists.

Moreover [illegible] with respect to x

[illegible]

or, by using (4),

(5_1) $f''(x+h)(1-\xi') - f''(x+\xi)(h\xi'' + \xi') = 0$,

and [illegible] by further differentiation with respect to x

(5_2) $f'''(x+h)(1-\xi') - f''' (x+\xi)(h\xi'' + \xi') = 0$.

Now, however, as [illegible] clear [illegible] the determinant of the two equations (5_1) and (5_2)

$$\begin{vmatrix} f''(x+h) & f''(x+\xi) \\ f'''(x+h) & f'''(x+\xi) \end{vmatrix} = 0.$$

[illegible] then [illegible] the previous considerations does not happen. [illegible] of the equation [illegible] the above [illegible] intermediate equation admits of being written in the form

$$\frac{f'''(x+h)}{f''(x+h)} = \frac{f'''(x+\xi)}{f''(x+\xi)}.$$

Since [illegible] $\theta < 1$ [illegible] may put

$$\xi = h + p\,h$$

where [illegible]. Therefore [illegible] with [illegible] changes to

$$\frac{f'''(x)}{f''(x)} = \frac{f'''(x+p)}{f''(x+p)}.$$

But p cannot be an [illegible] constant [illegible] and therefore by [illegible] which would again lead to the case

had the period [illegible] would also have [illegible] By repeatedly proceeding as above it would follow that [illegible] has an arbitrarily small period and must therefore be constant. [illegible] follows [illegible] solution.

of [illegible] $f(x$ [illegible] therefore $y = f$ [illegible] and [illegible] differentiable and therefore [illegible] the same [illegible] for [illegible] $h)$ and therefore [illegible] can take all the values of a certain [illegible]. This leads however with respect [illegible] equation [illegible] the

result that $\frac{f'''(x)}{f''(x)}$ must be constant. If [illegible] writes [illegible] in the [illegible]

$$\frac{f'''(x)}{f''(x)} = C$$

where C denotes a constant [illegible] this equation [illegible] determined:

$$[illegible] \left([illegible] \right) - \frac{C_2}{C} \quad \text{for } C \neq 0,$$

$$f(x) = C_1 \frac{x^2}{2} + C_2 x + C_3 \quad \text{for } C = 0;$$

C_1, C_2, C_3 being constants.

Leaving aside the [illegible] of [illegible] which gives θ independent of h, we have the expression [illegible] with [illegible]

17. *Theorem III.* Of [illegible] the functions [illegible] which are continuous in (a, b), the only [illegible] being [illegible] and [illegible] being excluded, there is no function for which the theorem M_1 is satisfied with a

θ independent [illegible]

Proof: "The left side of the equation

$$\frac{f(x+h)-f(x)}{h} = f'(\xi), \; (\xi = x + \theta h)$$

is for $h \neq 0$ and for every fixed value of x in the interval $a < x < b$ continuous and differentiable with respect to h [illegible] since the argument ξ is a linear function of h [illegible] differentiable with respect to h for every fixed value of x, the [illegible] the same [illegible] f [illegible] exists in the interval $a < x$ [illegible]. Moreover [illegible] of (M_1) with respect to h and with respect to x gives the two equations

$$f'(x+h) = f'(x+\theta h) + hf''(x+\theta h)\,\theta'$$

$$f'(x+h) - f'(x) = h\, f''(x+\theta h)(1+\theta' h)$$

From the first follows at once, because of [illegible] θ, that for h [illegible] also $f'(\xi)$ is continuous, [illegible] $f'(x)$ is continuous for $a < x < b$.

In the second equation the factor of [illegible] $(x + \theta h)$ cannot vanish for every value [illegible] consequently [illegible] θ' admits of being eliminated and one obtains thus

$$(1 + \theta' h - \theta) f'(x + h) + \theta f'(x) = (1 + \theta' h) f'(x + \theta h).$$

One more differentiation with respect to h gives

$$\theta' f'(x + h) + (1 + \theta' h - \theta) f''(x + h) = \theta' f'(x + \theta h) + (1 + \theta' h) f''(x + \theta h) \theta$$

Now [illegible] converge to [illegible]. Thereby one obtains by having regard to the continuity of $f''(x)$

$$(1 - \theta) f''(x) = f''(x) \theta$$

and since the identical vanishing of [illegible] would lead to linear $f(x)$, there remains the only possibility

$$\theta = \tfrac{1}{2}$$

i. e., θ is the same constant as in Theorem 1.**

§ 42

[illegible] now be given briefly [illegible] treatment of the question What conditions must be satisfied by θ as a function of x and h in order that there should be a differentiable function [illegible] to satisfy the mean-value theorem

$$f(x + h) = f(x) + h f'(x + \theta h)?$$

Let θ be a function, subject to the condition $0 < \theta < 1$, of the two independent variables x_1 [illegible] $x_2 = x + h$, which take values inside the quadratic region

$$a < x_1 < b, \quad a < x_2 < b.$$

The function $\xi = x_1 + \theta(x_1, x_2)(x_2 - x_1)$ takes only values of the interval $a < \xi < b$. The problem is: What conditions must be satisfied by ξ as a function of x_1, x_2 in order that there be a function $f(x)$ which is continuous for $a \leq x \leq b$, differentiable for $a < x < b$ and satisfies the equation

(M) $\quad f(x_2) - f(x_1) = (x_2 - x_1) f'(\xi)$?

Let those pairs x_1, x_2 be excluded for which the following conditions I – IV are not satisfied and let the remaining part of the region be denoted by R:

I. $x_1 \neq x_2$ [illegible] $h = x_2 - x_1 \neq 0$

II. $\theta(x_1, x_2)$ possesses the continuous partial differential co-

...cients $\frac{\partial\theta}{\partial x_1}=\theta_1$, $\frac{\partial\theta}{\partial x_2}=\theta_2$, $\frac{\partial\theta_1}{\partial x_2}=\frac{\partial\theta_2}{\partial x_1}$ ... $\frac{\partial\theta}{\partial x}$ ... $=\theta_{12}$ ... $\frac{\partial\theta_{12}}{\partial x_2}$

$=\theta_{12}$ ...

It is easily seen that the function [illegible] is represented by $\xi(x_1, x_2)$. The partial differential coefficients $\frac{\partial\xi}{\partial x_1}=\xi_1$, $\frac{\partial\xi}{\partial x_2}=\xi_2$ are not [illegible] identically 0, for otherwise ξ would be a constant and therefore [illegible] which would give according to (M) a linear $f(x)$ which case was excluded beforehand.

Now the left side of (M) and consequently also the right side are partially differentiable with respect to x_1 as well as x_2 and because by II, ξ_1 and ξ_2 are existent therefore [illegible] exists and is equal $\frac{\partial f(\xi)}{\partial x_1}\Big/\xi_1=\frac{\partial f(\xi)}{\partial x_2}\Big/\xi_2$, it being assumed that ξ_1 and ξ_2 are both different from zero. Under the supposition by the partial differentiation of (M) we have

$$(M_1)\left\{\begin{array}{l} f(x_2) = (x_2-x_1)f'(\xi)\,\xi_2+f(\xi) \\ -f(x_1) = (x_2-x_1)f'(\xi)\,\xi_1-f(\xi) \end{array}\right.$$

and hence because of I

$$f'(\xi)=\frac{f(x_2)-f(\xi)}{(x_2-x_1)\,\xi_2} \quad \frac{f(x_1)-f(\xi)}{[illegible]}$$

III. Neither ξ_2 nor ξ_1 vanish.

The expression for $f'(\xi)$ ... limits of partial differential coefficients in the whole of the region (B) and because of

$$f'(\xi)=\frac{\partial f(\xi)}{\partial x_1}\Big/\xi_1 \quad \frac{\partial f(\xi)}{\partial x_2}\Big/\xi_2$$

$f''(\xi)$ is existent in the whole of (B). Now then [illegible] differentiating with respect to x_1 both the sides of the first equation (M) or with respect to x_2 both the sides of the second equation we have

(M'_1) $(x_2-x_1)f''(\xi)$ ... $\xi_{12}+(x_2-x_1)f''(\xi)$ ... $\xi_1\xi_2+f'(\xi)$ ... $\xi_2=0$

where

$$\xi_{12}=\frac{\partial\xi_1}{\partial x_2}=\frac{\partial\xi_2}{\partial x_1}$$

IV. $f''(\xi) \neq 0$.

[illegible]

$$f''(\xi)(\xi_1\dot{\xi}_2 - \xi_2\dot{\xi}_1) = 0$$

[illegible]

Let θ be given by

$$[illegible] \qquad \ldots \quad (1)$$

[illegible] that

$$[illegible] \qquad (2)$$

[illegible] it is clear that [illegible] $f(x)$ we must have

[illegible]

[illegible] where [illegible]

[illegible] the form

[illegible]

§ 48.

70. The following theorems, the proof of each of which is briefly indicated, may be considered to be generalisations of the mean-value theorem in the sense that each of them depends for its validity on Rolle's theorem and gives the mean value theorem as a particular case.

(a) *Lagrange's remainder theorem*

$$f(x+h)=f(x)+hf'(x)+\dots+\frac{h^{n-1}f^{(n-1)}(x)}{(n-1)!}+\frac{h^n f^{(n)}(x+\theta h)}{n!},\quad 0<\theta<1,\ n\geq 1.$$

Let $\psi(t)$ denote $f(b)-f(t)-(b-t)f'(t)-\dfrac{(b-t)^2}{2!}f''(t)$

$$-\dots-\frac{(b-t)^{n-1}}{(n-1)!}f^{(n-1)}(t)-\frac{(b-t)^n}{n!}\mathrm{P},$$

where $b=x+h$ and P is independent of t and is given by

$$f(x+h)=f(x)+hf'(x)+\dots+\frac{h^{n-1}f^{(n-1)}(x)}{(n-1)!}+\frac{h^n}{n!}\mathrm{P}.$$

Then $\psi(x)=0$, $\psi(x+h)=0$ and if $f^{(n)}(t)$ exists for every value of t inside $(x, x+h)$, Rolle's theorem gives

$$\psi'(t)=0$$

for a value of t between x and $(x+h)$. But

$$\psi'(t)=\frac{(b-t)^{n-1}}{(n-1)!}\{\mathrm{P}-f^{(n)}(t)\}.$$

Therefore we have

$$\psi'(x+\theta h)=0,\ i.e.,\ \mathrm{P}=f^{(n)}(x+\theta h).$$

(b) *Cauchy's generalised mean value theorem*

$$\frac{\phi(x+h)-\phi(x)}{\mathrm{F}(x+h)-\mathrm{F}(x)}=\frac{\phi'(x+\theta h)}{\mathrm{F}'(x+\theta h)},\ 0<\theta<1.$$

Let

$$\psi(t)=\phi(t)-\phi(x)-\frac{\phi(x+h)-\phi(x)}{\mathrm{F}(x+h)-\mathrm{F}(x)}\{\mathrm{F}(t)-\mathrm{F}(x)\},$$

then $\psi(x)=0$, $\psi(x+h)=0$; therefore Rolle's theorem gives

$$\psi'(t)=0$$

for a value of t between x and $x+h$ if $\phi'(t)$ and $\mathrm{F}'(t)$ exist for every value of t inside $(x, x+h)$, and $\mathrm{F}'(t)$ is nowhere 0 or infinite. Hence the theorem.

(c) *Generalised mean value theorem*

$$\begin{vmatrix} f(x+h) & \phi(x+h) & \mathrm{F}(x+h) \\ f(x) & \phi(x) & \mathrm{F}(x) \\ f'(x+\theta h) & \phi'(x+\theta h) & \mathrm{F}'(x+\theta h) \end{vmatrix}=0,\ 0<\theta<1.$$

Let

$$\psi(t) = \begin{vmatrix} f(x+h) & \phi(x+h) & F(x+h) \\ f(x) & \phi(x) & F(x) \\ f(t) & \phi(t) & F(t) \end{vmatrix},$$

then [illegible] therefore [illegible] the [illegible] given

$$\psi'(t) = 0$$

[illegible] exist for every value [illegible] Hence the theorem [illegible] may [illegible] generalized theorem and [illegible] ordinary [illegible]

[illegible] functions [illegible] we have

$$\begin{vmatrix} [illegible] & f_1^{(n-1)}(u) & \dots & \dots & f_n^{(n-1)}(u) \\ f_0(x_1) & f_1(x_1) & \dots & \dots & f_n(x_1) \\ f_0(x_2) & f_1(x_2) & \dots & \dots & f_n(x_2) \\ \vdots & & & & \\ f_0(x_n) & f_1(x_n) & & & f_n(x_n) \end{vmatrix} = 0,$$

where u is a mean of $x_1, x_2, \dots x_n$.

(d) *Pompeiu's theorem* [illegible]

[illegible] $0 < \theta < 1$

The theorem [illegible] that ϕ and F are positive and increasing.

[illegible]

$0 < \theta < 1$;

$\phi(u, v)$ standing for

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Thus

$$\chi'(t) = h\phi(x + th, y + tk) + k\psi(x + th, y + tk) \text{ [illegible]}$$

(f) *Goursat and Hedrick's form** [illegible]

$$f(x+h, y+k) = f(x, y) + hf_x(x+\theta h, y) + kf_y(x, y+\theta k) \text{ [illegible]}$$

follows by considering

$$\chi(t) = f(x + ht, y + k) + f(x, y + kt)$$

§ 44

71. The number θ which occurs in any of the generalized theorems of § 43 can be studied in the same manner as the θ which occurs in the ordinary mean value theorem; I will conclude this lecture with a brief account of the researches relating to these θ's.

(1) The number θ in Lagrange's remainder was probably studied by Cauchy and his school with the result that θ [illegible] was shown to be $\frac{1}{n+1}$, but the first published investigation about θ is due to [illegible][†] who gave the first four terms in the expansion of θ(h), these being given as follows,

$$\theta = \frac{1}{n+1} + h\left[\left\{\frac{2(n+1)-(n+2)}{2(n+1)^2(n+2)}\right\}\frac{f^{(n+2)}(x)}{f^{(n+1)}(x)}\right]$$

$$+\frac{h^2}{2}\left[\left\{\frac{2\cdot 3\,(n+1)^2-(n+2)(n+3)}{3\,(n+1)^3(n+2)(n+3)}\right\}\frac{f^{(n+3)}(x)}{f^{(n+1)}(x)} \text{ [illegible]}\right.$$

$$\left.-\left\{\text{[illegible]}\right\}\left\{\text{[illegible]}\right\}\right]$$

$$+\frac{h^3}{3!}\left[\left\{\text{[illegible]}\right\}\text{ [illegible]}\right.$$

$$\left\{\text{[illegible]}\right\}$$

$$\left\{\frac{f^{(n+2)}(x)}{f^{(n+1)}(x)}\cdot\frac{f^{(n+3)}(x)}{f^{(n+1)}(x)}\right\}$$

$$\left\{\text{[illegible]}\right\}$$

$$\left.-\left\{\text{[illegible]}\right\}\left\{\text{[illegible]}\right\}\right]-$$

* *Mathematical Analysis*, Vol. I.

† [illegible]

(a) As stated in the first lecture, Rothe and Hayashi have studied this θ and Rothe has given a number of results similar to those for Rolle's function θ.

(ii) [illegible] has shown that if [illegible] exists, is finite and $\neq 0$, $\theta(+0)$ exists and equals $\frac{1}{n+1}$.

(b) (i) The number θ occurring in Cauchy's generalized theorem was apparently studied at Cambridge as early as 18?4 for which year there is a question paper* containing the question

If [illegible] denote [illegible] prove that

$$\theta = \frac{1}{2} + \frac{h}{24}\,[illegible]\text{ approximately when } h \text{ is small.}$$

In the above θ is given by

$$\frac{f(x+h)-f(x)}{\phi(x+h)-\phi(x)} = \frac{f'(x+\theta h)}{\phi'(x+\theta h)}.$$

(ii) As Rothe considered the case in which Rolle's function $\theta(x, h)$ is an absolute constant, so Takahashi has considered the case in which the θ in Cauchy's generalized theorem is an absolute constant.[2] He has proved that θ must be $\frac{1}{2}$ and that, f and ϕ being assumed to be differentiable five times, they must belong to only the following three types:

(1) $f = Ax^2 + Bx + c, \quad \phi = A'x^2 + B'x + c'$

(2) $f = Ae^{px} + Be^{-px} + c, \quad \phi = A'e^{px} + B'e^{-px} + c'$

(3) $f = A \sin(px+q) + B, \quad \phi = A' \sin(px+r) + B'$

or

$A \cos(px+q) + B, \quad A' \cos(px+r) + B'$

In the above $p = 0$ in (1), $p > 0$ in (2) and $p < 0$ in (3); all the other constants are arbitrary but different from 0.

(c) Takahashi has considered a question similar to the above for the θ in Genocchi and Peano's theorem.[3]

[1] [illegible]

* Of Trinity College, June [illegible]

[2] l.c., pp. [illegible]

[3] l.c., pp. [illegible]

APPENDIX A.

On Pompeiu's Proof of the Mean Value Theorem

UNIVERSITÉ DE BUCAREST,
le 15 Mars, 1931.

Très honoré Monsieur et Professeur,

Je viens de recevoir votre lettre ainsi que l'extrait de votre livre où (§ 15) vous me faites l'honneur de donner une place à ma méthode pour démontrer le théorème des accroissements finis.

Veuillez en accepter tous mes remerciements très sincères.

Vous faites suivre l'exposé de ma démonstration de quelques observations (criticism of Pompeiu's proof) sur lesquelles vous désirez avoir aussi mon opinion.

Je suis très sensible à la délicatesse de ce procédé.

Voici quelle est mon opinion.

Le *fait* que vous signalez par votre exemple (page 28) est *exact* et, de plus, c'est un *fait général* dans le cas d'une fonction dérivée quelconque, c'est-à-dire pourvue de points de discontinuité.

Mais ce fait n'intervient pas dans ma démonstration.

Dans ma démonstration le point c est toujours *intérieur* aux intervalles (x_1, y_1) et alors le fait signalé par votre exemple *ne peut pas se produire*.

Dans votre exemple le point $c = \frac{1}{2}$ est *extérieur* aux intervalles

$$x_1 = \tfrac{1}{2} + \frac{1}{2\pi n}, \quad y_1 = \tfrac{1}{2} + \frac{1}{2\pi n + \frac{\pi}{2}},$$

c'est pour cela que le fait signalé par vous est possible.

En résumé, vous signalez par votre exemple un fait *exact*, mais ce fait n'a rien à voir dans ma démonstration, parce que les circonstances sont autres dans ma méthode: le point c est toujours *intérieur* aux intervalles (x_1, y_1) et alors le fait en question ne peut pas se produire.

Et, puisque, ainsi éclaircie, la question présente un intérêt scientifique, j'ai rédigé une Note (ci-jointe) et, si vous trouvez cela convenable, vous pouvez (après traduction en anglais, ou même sous sa forme primitive) la faire figurer à la fin de votre livre comme une simple Note explicative.

Je vous en serais très obligé pour l'intérêt accordé à la question.

Veuillez agréer, très honoré Monsieur le Professeur, l'assurance de ma considération très distinguée et dévouée.

votre tout dévoué

D. Pompeiu

Agrégé à l'Université de Paris

NOTE

Sur une propriété des fonctions dérivées

par D. Pompeiu.

1. Soit $f(x)$ une fonction définie dans un intervalle (a, b) et admettant, pour tout point x, intérieur à (a, b), une dérivée bien déterminée $f'(x)$.

Cela veut dire que, si x est un point fixe pris dans (a, b), le rapport

$$\frac{f(x+h)-f(x)}{h}$$

a une limite bien déterminée lorsque h tend vers zéro (par valeurs positives ou négatives).

2. Considérons maintenant c un point pris dans l'intérieur de (a, b) et (x_n, y_n) une suite d'intervalles contenant c, de plus en plus petits et tendant vers le point c.

Formons les rapports

$$\frac{f(x_n)-f(y_n)}{x_n-y_n} = R(x_n, y_n).$$

Que peut-on dire de la limite des rapports $R(x_n, y_n)$, lorsque les intervalles (x_n, y_n) tendent vers le point c?

Voici la réponse précise :

1° Cette limite peut ne pas exister du tout. On peut facilement former un exemple avec la fonction dérivée

$$f'(x) = \sin\frac{1}{x}$$

dans le voisinage du point $x = 0$.

2° Cette limite peut exister et être différente de la valeur de $f'(x)$ au point $x = c$.

C'est l'exemple signalé à la page 24 de ce livre.

3° Cette limite existe et est égale à la dérivée $f'(c)$ si tous les intervalles (x_n, y_n) contiennent le point c à leur intérieur.

[illegible] immédiatement de l'identité

$$\frac{f(x_2)-f(y_2)}{x_2-y_2}=\frac{f(x_2)-f(c)}{x_2-c}\cdot\frac{x_2-c}{x_2-y_2}+\frac{f(c)-f(y_2)}{c-y_2}\cdot\frac{c-y_2}{x_2-y_2}.$$

Puisque c est intérieur à (x_2, y_2) on a

$$\left|\frac{x_2-c}{x_2-y_2}\right|<1,\quad \left|\frac{c-y_2}{x_2-y_2}\right|<1$$

et alors le rapport du premier membre est compris entre les deux rapports qui figurent au second membre.

C'est ce fait seul qui se présente dans la démonstration du théorème [illegible] exposée aux pages 2[illegible]–27 [illegible].

3. Mais il résulte de ce qui précède que la dérivée d'une fonction $f(x)$ peut être définie

soit par la limite du rapport classique

$$(1)\qquad \frac{f(x+h)-f(x)}{h},$$

soit par le rapport

$$(2)\qquad \frac{f(x')-f(x'')}{x'-x''}$$

avec la condition expresse que le point x où l'on veut avoir la dérivée soit constamment compris entre x' et x''

$$x' < x < x''.$$

Alors la dérivée cherchée est aussi bien la limite du rapport (1) que la limite du rapport (2).

D. Pompeiu

Remarks on the above

By

Ganesh Prasad

(I)

The fact that $\frac{f(x_2)-f(y_2)}{x_2-y_2}$ tends to a limit as x_2 and y_2 both tend to [illegible] even if, for every value of [illegible],

$$x_2 < c < y_2,$$

does not carry us far, as will be clear from the following examples, in each of which [illegible].

Example 1. Let $f(x) =$ [illegible] and take $x_2 = -y_2$, x_2 tending to [illegible] in any manner whatever.

Then $f(x_1) - f(y_1)$ is zero for every value of n; consequently x_1 and y_1 both tend to 0 while $\dfrac{f(x_1)-f(y_1)}{x_1-y_1}$ also tends to 0.

But there is no differential coefficient for $f(x)$ at $x=0$, and as a matter of fact the mean-value theorem does not hold.

Example 2. Let $f'(x) = \sin\frac{1}{x}$, so that

$$f(x)=\int_0^x \sin\frac{1}{t}\,dt;$$

and take x_1, [illegible] y_1, x_1 tending to 0 in any manner whatsoever.

Contrary to the statement under 1 [illegible] $f'(0)$ exists[1] and is 0.

Also $f(x_1)$ [illegible] y_1) [illegible] is an even function; consequently x_1 and y_1 both tend to 0 while $\dfrac{f(x_1)-f(y_1)}{x_1-y_1}$ also tends to 0.

Also there is a differential coefficient for $f(x)$ at $x=0$, and as a matter of fact the mean-value theorem holds.

Example 3. Let $f(x) = x\sin\frac{1}{x}$, and take $x_1 = -y_1$, x_1 tending to 0 in any manner whatsoever.

Then $f(x)$ being an even function, $f(x_1) - f(y_1)$ is zero for every value of h; consequently x_1 and y_1 both tend to 0 while

$$\frac{f(x_1)-f(y_1)}{x_1-y_1}$$

also tends to 0.

Although there is no differential coefficient at $x=0$ the mean-value theorem holds.*

[1] This may be proved as follows: [illegible]

$$[illegible]$$

Therefore [illegible] the sum of the differential coefficients [illegible]

$$[illegible]$$

[illegible] coefficient 0 at $x=0$.

* See [illegible]

(II)

Leaving [illegible] the Note of Professor Pompeiu [illegible] in this Appendix and going back [illegible] proof [illegible] 15 [illegible] with [illegible] that like the classical proof of [illegible] is based on the following property [1] of continuous functions:

If the function [illegible] is continuous between [illegible] and [illegible] and in the points a and [illegible] of this interval [illegible] takes different values A and B, then for one or more determinate values of x between a and [illegible] shall take any value C comprised between A and B.

For the purposes of [illegible] proof, the above property will [illegible] with the ordinary [illegible] continuity in which [illegible] but with [illegible] Professor Pompeiu's [illegible] 15 [illegible] different [illegible]

[illegible] the purposes of [illegible] proof [illegible] $f(x)$ [illegible] Professor Pompeiu's proof

$$R(x, a_1) = \frac{f(x) - f(a_1)}{x - a_1}$$

[illegible] convention that [illegible] a_1) [illegible] opinion that [illegible] proof [illegible] to the [illegible] entirely unjustifiable.

This opinion [illegible] in conformity with the view held at one time by Professor Pompeiu [illegible] paper [2] published after the publication of the paper [illegible] proof of 15. Professor Pompeiu says: "[illegible] the proof which [illegible] given [illegible] property of continuous functions. Now this property has been extended in the present paper to generalised continuous functions and [illegible] proof applicable to the general case. In fact [illegible] the ratio

$$R(x, a) = \frac{F(x) - F(a)}{x - a}$$

and complete its definition by putting

$$R(a, a) = F'(a)$$

[illegible] may [illegible] at the point $x = a$, [illegible] continuous (in the extended sense) at the point a."

[1] [illegible] Fondamenti per [illegible] p. [illegible]

[2] Sur [illegible] *de l'Université de Jassy*, 1908.

APPENDIX B

[illegible]

The [illegible] [illegible] value [illegible] [illegible] the mean value theorem [illegible] in this Appendix first [illegible] [illegible]

List of the Various Forms

I. [illegible] $\phi'(a+\theta h)$, $0<\theta<1$.

[illegible]

II. [illegible]

[illegible] also *Oeuvres*, (2) T. IV, p. 200.]

III. [illegible]

[illegible] *Handbuch* [illegible]

IV. [illegible]

V. [illegible] any continuous function [illegible] $a<t<b$ [illegible]

[illegible] Art. [illegible]

[illegible]

V. [illegible]

[illegible] any [illegible]

[illegible]

VI. [illegible]

[illegible] 1881, p. [illegible]]

VII. [illegible]

[Lacroix, *Traité* [illegible] *intégral*, 2nd edition, 1800, p. 399.]

VIII. [illegible]

[Lacroix, *ibid.*, p. 400.]

IX. [illegible]

[Laplace, *Théorie* [illegible]. Also *Oeuvres*, t. VII, p. 179.]

X. $\frac{1}{n!}\int_0^h (h-t)^{n-1} f^{(n)}(x+t)\,dt$ [illegible]

[Lacroix [illegible], 2nd edition [illegible]]

Brief Historical Remarks on the [illegible]

1. The term [illegible] It is found in his *Leçons sur le calcul des fonctions* [illegible] polytechnique in 1799 and published first in 1801 in the *Recueil des leçons de l'école normale*, and again printed in 1806 in the *Journal de l'école polytechnique*. It may however be traced in his *Théorie des fonctions analytiques*, published in 1797 [illegible] form of a definite integral [illegible] equivalent [illegible]

[illegible] and contents himself [illegible] the working will do for any n.

[illegible] Cauchy, who appears to [illegible] the form [illegible]

[illegible] pointed [illegible] [illegible] *Journal* [illegible] *Handbuch* [illegible] following form of the remainder

$$[illegible]$$

where ψ [illegible] arbitrary function [illegible] to the conditions that [illegible] between the [illegible]

[illegible]

Liouville's Journal on pp. 271–272.

It may be added [illegible] method used by [illegible] mean-value theorem [illegible] deduces his result from the form

$$\frac{1}{(n-1)!}\int_0^h t^{n-1}f^{n}(x+h-t)\,dt$$

of the remainder.

[illegible] word for word.

[illegible] known the [illegible] ratio of two functions

$$\frac{F(a+h)-F(a)}{[illegible]}=\frac{F'(a+\theta h)}{[illegible]} \qquad (\,)$$

where [illegible] F and [illegible] are supposed to be continuous and [illegible] to be a function which never vanishes between a and $a+h$. Let then in fact

$$[illegible] = f(x+h) - f(x) - (a+h-x)f'(x) - \cdots - \frac{(a+h-x)^n}{n!}f^n(x)$$

$$[illegible] = \frac{h}{q} [illegible]$$

[illegible] *Messenger of Math.* Vol. [illegible] pp. [illegible]

whence

$$F(x) = [illegible] \qquad F^{n}([illegible]) = [illegible]$$

Let us assume that the functions f, ϕ and their [illegible] remain finite and continuous in the interval from a to $a+h$ and that $\phi^{(n+1)}(x)$ is not zero; the same conditions are [illegible] the functions F and Φ, and one may apply to them the formula (1) which reduces now in this case to

$$\frac{F(a)}{\Phi(a)} = \frac{F^{n}(a+\theta h)}{\Phi^{n}(a+\theta h)},$$

because

$$F(a+h) = 0, \quad \Phi(a+h) = 0.$$

Hence follows at once

$$\frac{f(a+h)-f(a)-hf'(a)-\cdots-\dfrac{h^n}{n!}f^{(n)}(a)}{\phi(a+h)-\phi(a)-h\phi'(a)-\cdots-\dfrac{h^n}{n!}\phi^{(n)}(a)} = \frac{f^{(n+1)}(a+\theta h)}{\phi^{(n+1)}(a+\theta h)}, \qquad (2)$$

a relation of which the formula (1) is a particular case.

If now one puts

$$f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + R,$$

the equation (2) may be written

$$R = \left\{\phi(a+h) - \phi(a) - h\phi'(a) - \cdots - \frac{h^n}{n!}\phi^{(n)}(a)\right\} \frac{f^{(n+1)}(a+\theta h)}{\phi^{(n+1)}(a+\theta h)};$$

by giving to the arbitrary function ϕ such forms as one pleases (satisfying the conditions enunciated above) one shall have all the expressions for the remainder in Taylor's series. The equation (3) is therefore the general form of this remainder.

For example, if one puts

$$\phi(x) = (x-a)^{[illegible]},$$

one finds

$$R = [illegible]$$

where [illegible] and [illegible] are undetermined [illegible] integral [illegible] than [illegible] positive number [illegible] very [illegible] expressions for the remainder.

In particular, for $p = q$,

$$R_n = \frac{h^{n+1}(1-\theta)^{n-p}}{n!(p+1)} f^{n+1} [illegible]$$

a form [illegible] which I have given [illegible] for representing [illegible] between [illegible]. In fact [illegible] the ordinary remainder and the [illegible] by

When one puts [illegible] in the general expression [illegible] formula of Schlömilch

[illegible]

[illegible] in the interval [illegible] in other words if in the interval [illegible] varies always [illegible] and then

[illegible]

which [illegible] for the remainder independent [illegible]

[illegible] In [illegible] Giuseppe Peano [illegible] notes before the [illegible] text. These notes are by Prof. Peano and [illegible] from [illegible] to [illegible] the page [illegible] among the remarks in Art. 67 [illegible] of the text appear the following lines which because of their historical importance are reproduced here [illegible] word for word in English translation:

The formula of Taylor may be [illegible] in the following manner.

If there exist the 1st, 2nd, [illegible] differential coefficients of [illegible] for $x = x_0$, then

[illegible]

where [illegible] is a quantity which has for [illegible] with h tending to [illegible]. If we put $n = 1$, then we have the formula which serves for the [illegible]

of the differential coefficient. Of course, if $f(x)$ has the nth differential coefficient for $x = a$, then it ought to have the preceding differential coefficients also in the neighbourhood of $x = a$, but as to the nth differential coefficient, it is not necessary to suppose either the existence or the continuity in the neighbourhood of x.

As stated on p. 70 of this book, the form VI is also given in Dini's *Calcolo infinitesimale*, t. 1 (1907), which practically contains the subject matter of Dini's lectures of much earlier years. It is therefore unfortunate that recently two books[1] have come out in English in which the remainder VI is prominently described as Young's form of the remainder. No doubt this mistake is due to the ignorance of the authors of the works of Peano and Dini, and may be excused, especially in view of the fact that so well-informed a mathematician as the famous W. H. Young gives the remainder in his book *Fundamental Theorems of the Differential Calculus* (1910) without mentioning Dini or Peano.

VII. [illegible] the discoverer of this form [illegible] D'Alembert, as is evidenced by [illegible] remark[2]: [illegible] D'Alembert has [illegible] and demonstrated [illegible] the same time the theorem of Taylor (*Recherches sur différents points* [illegible] *du système du monde*, t. 1 [illegible] p. [illegible]).

That Lacroix is wrong is the opinion of Professor Pringsheim[3] who says: "A previously given derivation of Taylor's series by D'Alembert shortly before the publication of Euler's Differential Calculus is based on the (obviously inaccurate) relation[4]

$$f(x+h) = f(x) + \int f'(x+h)\,dh$$

and its repeated application to $f'(x+h)$, $f''(x+h)$, etc. With the repetition n times of this process of transformation would one attain to an expression for the remainder in Taylor's expansion in the form of a repeated integral with n integrations (of such a possibility has D'Alembert not even once explicitly said anything)."

[1] Phillips, *A Course of Analysis*, 1939;
[illegible], *Elementary Lessons in Analysis*, 1930.

[2] See Art. 384 of Lacroix's book.

[3] *L.c.*, p. 459.

[4] The correct relation is $f(x+h) = f(x) + \int_0^h f'(x+h)\,dh$.

VIII. (a) Although it is true that Lagrange was the first to give the remainder in the form of a definite integral, his form is fairly complicated and from it VIII can be derived only by a cumbersome process.

In fact Lagrange's statement giving his form of the remainder is included in the following [1] :—

Let

$$f(x+h)=f(x)+f'(x).\frac{h}{1!}+\cdots+f^{(n-1)}(x).\frac{h^{n-1}}{(n-1)!}+r_n(x,h).h^n,$$

then substitute $(x-h)$ for x and put

$$r_n(x-h,h)=q_n(x,h).$$

Thus

$$f(x)=f(x-h)+f'(x-h).\frac{h}{1!}+\cdots+f^{(n-1)}(x-h).\frac{h^{n-1}}{(n-1)!}+q_n(x,h).h^n. \qquad (1)$$

Put now

$$h=xz,\ q_n(x,xz).\ x^n=p_n(x,z),$$

then (1) gives

$$f(x)=f(x-xz)+f'(x-xz).\frac{xz}{1!}+\cdots+f^{(n-1)}(x-xz).\frac{x^{n-1}z^{n-1}}{(n-1)!}+p(x,z)z^n. \qquad (2)$$

Hence by partial differentiation with respect to z,

$$0=-f^{(n)}(x-xz).\frac{x^nz^{n-1}}{(n-1)!}+\frac{\partial p_n}{\partial z}.z^n,$$

i.e., $\dfrac{\partial p_n}{\partial z}=f^{(n)}(x-xz).\dfrac{x^{n-1}}{(n-1)!}.$

Now, from (2), we have $p_n(x,0)=0$. Therefore, finally,

$$p_n(x,z)=\frac{1}{(n-1)!}\int_0^z y^{n-1}f^{(n)}(x-xy)dy. \qquad (3)$$

If in (2) and (3) we write $\frac{t}{x}$ for z, we have

$$p_n\left(x,\frac{t}{x}\right)=\frac{x^n}{(n-1)!}\int_0^{\frac{t}{x}} y^{n-1}f^{(n)}(x-xy)dy$$

$$=\frac{1}{(n-1)!}\int_0^t t^{n-1}f^{(n)}(x-t)dt,$$

which is practically the same as any of the forms VIII—X.

[1] This is taken almost word for word from Pringsheim's paper (*l.c.*, pp. 441-442).

(b) Lacroix deduces VIII from VII by using the result

$$\int_0^h {}^{(n)} H(h) dh^n = \frac{1}{(n-1)!} \int_0^h (h-t)^{n-1} H(t) dt$$

(c) Laplace's derivation [1] of his form is substantially reproduced by Cauchy in his *Resumé* (*Leçon*, 36) and is as follows:—

"One has by taking the integral from $z=0$,

$$\int dz\phi'(x-z) = \phi(x) - \phi(x-z),$$

$$\int dz\phi'(x-z) = z\phi'(x-z) + \int z dz \phi''(x-z),$$

$$\int z dz\phi''(x-z) = \tfrac{1}{2} z^2 \phi''(x-z) + \tfrac{1}{2}\int z^2 dz \phi'''(x-z).$$

By continuing this process, one finds generally

$$\int dz\phi'(x-z) = z\phi'(x-z) + \frac{z^2}{1.2}\phi''(x-z) + \dots + \frac{z^n}{n!}\phi^{(n)}(x-z)$$

$$+ \int \frac{z^n dz}{n!} \phi^{(n+1)}(x-z).$$

By comparing this expression with

$$\int dz\phi'(x-z) = \phi(x) - \phi(x-z),$$

one shall have

$$\phi(x) = \phi(x-z) + z\phi'(x-z) + \frac{z^2}{2!}\phi''(x-z) + \dots$$

Putting $x-z=t$, the preceding equation takes the form

$$\phi(t+z) = \phi(t) + z\phi'(t) + \frac{z^2}{2!}\phi''(t) + \dots\dots + \frac{z^n}{n!}\phi^{(n)}(t)$$

$$+ \frac{1}{n!}\int_0^z z'^n dz' \phi^{(n+1)}(t+z-z')."$$

[1] *L.c.*

APPENDIX C.

Additions and Corrections.

Page 4.	Line 17:	*For*	" $f(x_0+h)\,f(x_0)$ "	*read*	" $f(x_0+h)-f(x_0)$."
Page 5.	2nd foot-note:	*For*	" 78 "	*read*	" 75."
Page 9.	Line 12:	*For*	"also nowhere"	*read*	"also generally nowhere."
Page 11.	Line 2:	*For*	"Pompieu"	*read*	"Pompeiu."
Page 11.	1st foot-note:	*For*	" theoreme "	*read*	" théorème."
Page 19.	Line 19:	*For*	" $'(x\phi+$ "	*read*	" $\phi'(x+$."
Page 19.	last foot-note:	*For*	" $x \triangle x$."	*read*	" $x \triangle x$."
Page 22.	Line 6 from bottom:	*For*	" $F'(\xi)$ "	*read*	" $F''(\xi)$."
Page 23.	3rd line from bottom:	*For*	" $\dfrac{\phi(x'-\delta)-\phi(x')}{\delta}$ "	*read*	" $\dfrac{\phi(x'-\delta)-\phi(x')}{-\delta}$ ".

Page 26. at the end: *Add as a foot-note to line 4*:

" Although, as shown in Appendix A, the statement in this sentence is correct, the example that follows is *not* to the point. Professor Pompeiu was good enough to respond to the author's request to express his opinion on the criticism and drew the author's attention to the second paper quoted on p. 97 in the 2nd foot-note. The proof, as modified in the light of this paper, is perfectly valid."

Page 30.	Line 20:	*For*	" $f'(\xi$ is "	*read*	" $f'(\xi)$ is."
Page 30.	foot-note:	*For*	" *Ibid* "	*read*	" See the foot-note on the preceding page."
Page 31.	Line 18:	*For*	" $f'\,b-0)$ "	*read*	" $f'(b-0)$."
Page 33.	Line 8 from bottom:	*For*	" ϕx "	*read*	" $\phi(x)$."
Page 44.	Line 1:	*For*	" inumerately "	*read*	" immediately."
Page 49.	Line 9:	*For*	" $2A_0A_1$ "	*read*	" $2A_0A_2$."
Page 51.	Line 3:	*For*	" $\{f^{(3)}\}^2 f^{(5)}$ "	*read*	" $\{f^{(3)}\}^2, f^{(5)}$."
Page 51.	Line 9:	*For*	" f^5 "	*read*	" $f^{(5)}$."
Page 53.	Line 2:	*For*	" left "	*read*	" right."
Page 53.	Line 11:	*For*	" (2) "	*read*	" (3)."
Page 53.	Line 16:	*For*	" equation and "	*read*	" equation by $f''(x)$ and."
Page 54.	Line 3:	*For*	" left "	*read*	" right."

Page 54.	last line:	*For*	" $hf' x+\xi)$ "	*read*	" $+hf'(x+\xi)$."
Page 60.	7th line from bottom:	*For*	" $\dfrac{\cos\psi\left(\frac{h}{2}\right)}{h^2\psi'\left(\frac{h}{2}\right)}$ "	*read*	" $\dfrac{\cos^2\psi\left(\frac{h}{2}\right)}{2h^2\psi'\left(\frac{h}{2}\right)}$ "
Page 62.	Add as foot-note to ξ_m in the last line: " ξ_m is the same as u_m."				
Page 64.	Line 10:	*For*	" difficult "	*read*	" not difficult."
Page 67.	Line 10:	*For*	" $x = \quad 2 \quad (4n+1)\pi$ "	*read*	" $x = \dfrac{2}{(4n+1)\pi}$ "
Page 68.	Line 7:	*For*	" the maxima "	*read*	" the proper maxima "
Page 68.	2nd line from bottom:	*For*	" 33 "	*read*	" 32."
Page 74.	Line 2:	*For*	" 32 "	*read*	" 31."
Page 75.	Foot-note:	*For*	" second paper "	*read*	" second and third papers."
Page 76.	*At the end of Art. 59 add*:				

The subject is carefully considered in Prasad's third paper (*Bulletin of the Calcutta Mathematical Society*, Vol. XXIII, pp. 57-66). No doubt, $\dfrac{d\xi}{dh}$ cannot be finite and different from zero or infinite with determinate sign. But there is some doubt if $\dfrac{d\xi}{dh}$ can be zero at a point h for which $f'(h)$ has a cusp; Prasad's opinion is that most probably $\dfrac{d\xi}{dh} = 0$ at such a point.

Page 80.	Line 1:	*For*	" values "	*read*	" of values."
Page 80.	Line 5 from bottom:	*For*	" $a_0\xi$ "	*read*	" a_0."
Page 83.	Line 15:	*For*	" $ce^{ax}\ \beta x$ "	*read*	" $ce^{ax}+\beta x$."
Page 85	Line 3 from bottom:	*For*	" h' "	*read*	" h."
Page 87.	Line 4:	*For*	" $\dfrac{\partial\xi}{\partial x}$ "	*read*	" $\dfrac{\partial\xi}{\partial x_1}$ "
Page 87.	Line 3 from bottom:	*For*	" $(x_2-x_1)f''(\xi).$ $\xi_1\ \xi_2$ "	*read*	" $(x_2-x_1)f''(\xi).$ $\xi_1\ \xi_2$."
		For	" $f''\xi(\xi_1-\xi_2)$ "	*read*	" $f''(\xi)(\xi_1-\xi_2)$."
Page 89.	Line 11 from bottom:	*For*	" $(Fx+h)$ "	*read*	" $F(x+h)$."
Page 90.	Line 5:	*For*	" h^n ! "	*read*	" h^n."